Triple Integrals

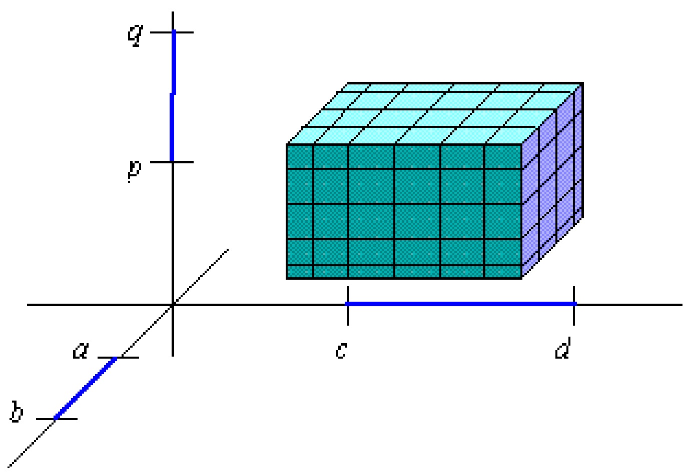
Part 1: Definition of the Triple Integral

We can extend the concept of an integral into even higher dimensions. Indeed, in this section we develop the concept of a *triple integral* as an extension of the double integral definition.

To begin with, suppose that $\phi(x, y, z)$ is a piecewise continuous function that assigns a number to each point in a solid Ω (the Greek capital "Omega"). Further, let us suppose that $\phi(x, y, z)$ is zero outside of Ω and that Ω is contained within a parallelpiped

$$[a, b] \times [c, d] \times [p, q] = \{(x, y, z) \mid a \leq x \leq b,\ c \leq y \leq d,\ p \leq z \leq q\}$$

(that is, $[a, b] \times [c, d] \times [p, q]$ is a "box").



A Riemann sum of $\phi(x, y, z)$ over tagged partitions $\{x_j, t_j\}_{j=1}^{m}$, $\{y_k, u_k\}_{k=1}^{n}$, and $\{z_l, v_l\}_{l=1}^{r}$ of $[a, b]$, $[c, d]$, and $[p, q]$, respectively, is a triple sum of the form

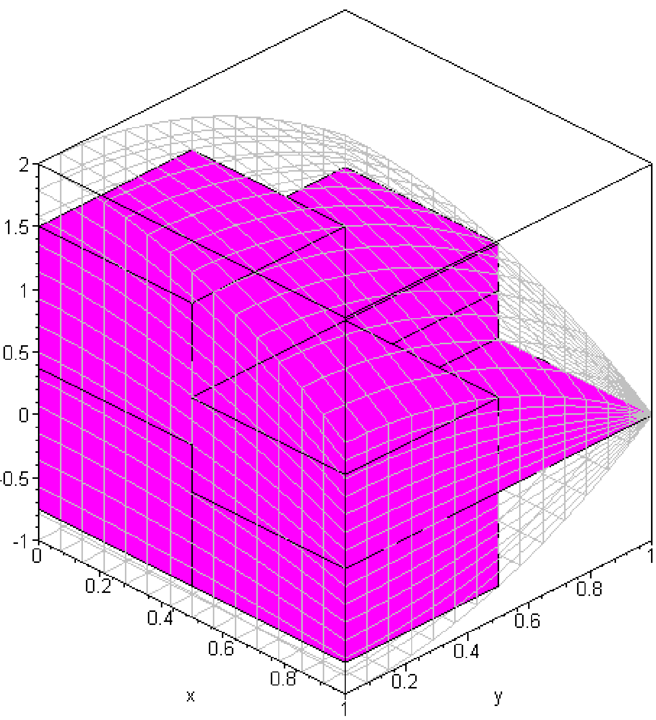
$$\sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{p} \phi(t_j, u_k, v_l) \Delta x_j \Delta y_k \Delta z_l$$

The *triple integral* of $\phi(x, y, z)$ over the solid Ω is the limit as h approaches 0 of Riemann sums over h-fine partitions:

$$\iiint_{\Omega} \phi(x, y, z)\, dV = \lim_{h \to 0} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{o} \phi(t_j, u_k, v_l) \Delta x_j \Delta y_k \Delta z_l$$

That is, the solid is "approximated" by a collection of "small boxes" with volume

$\Delta x_j \Delta y_k \Delta z_l$.



For example, if $f(x,y) \geq g(x,y)$ over a region R in the xy-plane, then the triple integral of $\phi(x,y,z)$ over the solid Ω bound between two surfaces $z = g(x,y)$ and $z = f(x,y)$ over the region R is given by

$$\iiint_{S_\Omega} \phi(x,y,z)\, dV = \iint_R \left[\int_{g(x,y)}^{f(x,y)} \phi(x,y,z)\, dz \right] dA_{xy} \tag{1}$$

where dA_{xy} is the area differential in the xy-plane.

EXAMPLE 1 Compute the triple integral of $\phi(x,y,z) = 8xyz$ over the solid between $z = 0$ and $z = 1$ and over the region

$$R: \quad \begin{array}{ll} x = 0 & y = 2 \\ x = 1 & y = 3 \end{array}$$

Solution: To do so, we use (1) to write

$$\begin{aligned}\iiint_\Omega 8xyz\, dV &= \iint_R \int_0^1 8xyz\, dz \;\; dA \\ &= \iint_R 4xyz^2\Big|_0^1 \;\; dA \\ &= \iint_R 4xy\, dA\end{aligned}$$

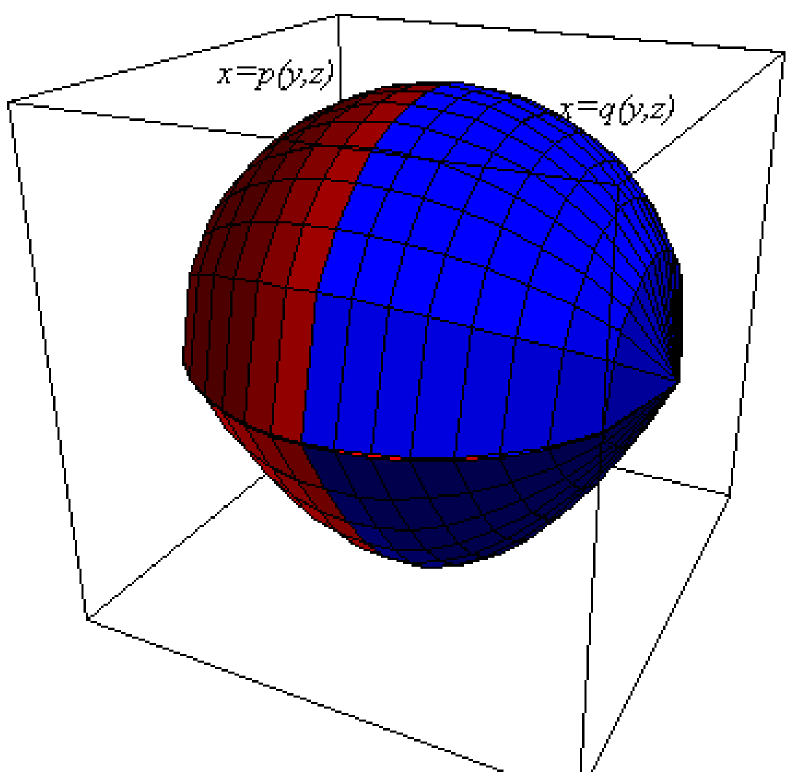
We then evaluate the resulting double integral over R:

$$\iiint_S 8xyz\, dV = \int_0^1 \int_2^3 4xy\, dydx = 5$$

A similar derivation to that above shows that the volume of Ω is given by

$$Volume\ of\ \Omega = \iiint_\Omega dV$$

(see the exercises). Moreover, in analogy with (1), if $p(y,z) \geq q(y,z)$ over a region R in the yz-plane,



then the triple integral of $\phi(x,y,z)$ over the solid Ω bound between the two surfaces $x = q(y,z)$ and $x = p(y,z)$ over the region S is given by

$$\iiint_\Omega \phi(x,y,z)\,dV = \iint_R \left[\int_{q(y,z)}^{p(y,z)} \phi(x,y,z)\,dx\right] dA_{yz}$$

where dA_{yz} is the area differential in the yz-plane.

EXAMPLE 2 What is the volume of the solid between $x = yz$ and $x = 0$ over the region $y = 0$, $y = 1$, $z = 0$, $z = 4$.

Solution: The volume is given by

$$V = \iiint_\Omega dV = \iint_S \int_0^{yz} dx\, dA_{yz}$$

Indeed, substituting the boundaries for Ω leads to the triple iterated integral

$$V = \int_0^4 \int_0^1 \int_0^{yz} dxdydz$$

Evaluating each integral in succession then leads to

$$V = \int_0^4 \int_0^1 yz\; dydz = \int_0^4 z\frac{y^2}{2}\bigg|_{y=0}^{y=1} dz = \int_0^4 \frac{z}{2} dz = 4$$

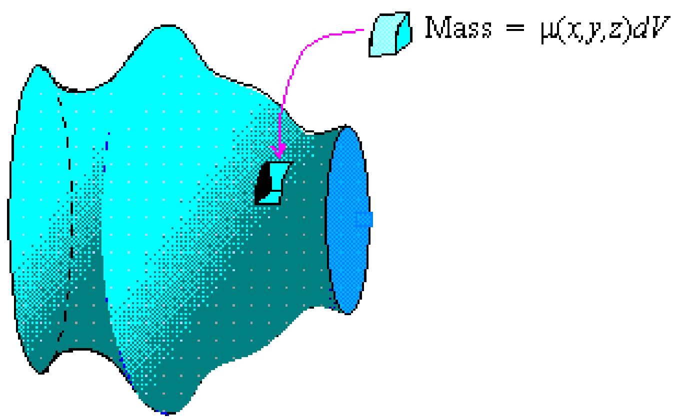
Check Your Reading: What type of solid is given in example 1?

Mass Density

In many applications, we are given a *mass-density*, $\mu(x,y,z)$, of a solid, which is a function measured in units of mass per unit volume (kg per m^3 or equivalently, grams per litre). Typically, if dM is the approximate mass of a small "chunk" of a solid with volume dV and if (x,y,z) is a point in that "chunk", then

$$dM = \mu(x,y,z)\,dV \tag{2}$$

which is to say that $\mu(x, y, z)\, dV$ is the mass of a small sample of a given solid.

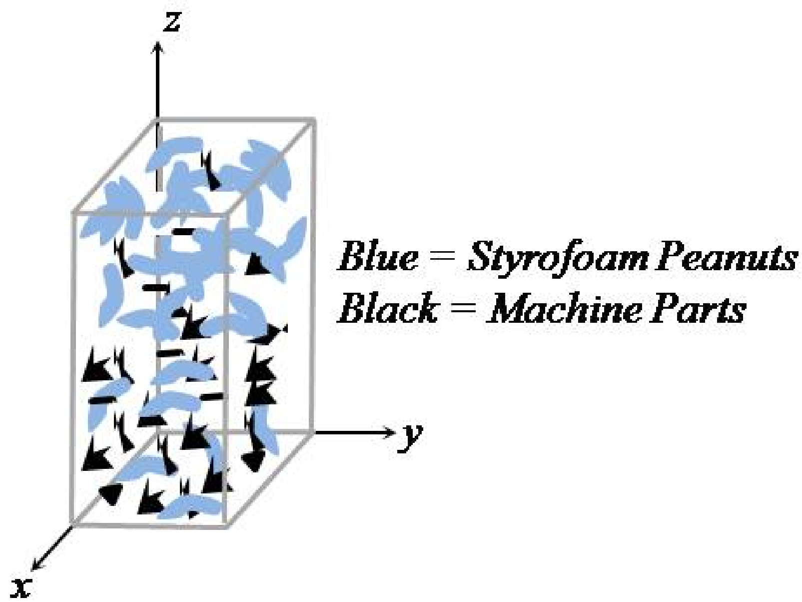


The definition of the triple integral then implies that the total mass of the solid S with mass density $\mu(x, y, z)$ is given by

$$M = \iiint_{\Omega} dM = \iiint_{\Omega} \mu(x, y, z)\, dV$$

That is, the mass of a solid depends on how dense it is.

EXAMPLE 3 A box corresponding to $[0, 1] \times [0, 1] \times [0, 2]$ in xyz-coordinates is filled with a mixture of small machine parts (heavy) and styrofoam peanuts (light), which via settling is heavier at the bottom than the top.



If the mixture has a density of

$$\mu(x, y, z) = \left(9 - z^3\right)\ \frac{kg}{m^3}$$

then what is the mass of the machine parts, styrofoam peanuts combination.

Solution: The mixture occupies the solid Ω corresponding to the interior of the box. Thus, the mass of the mixture is

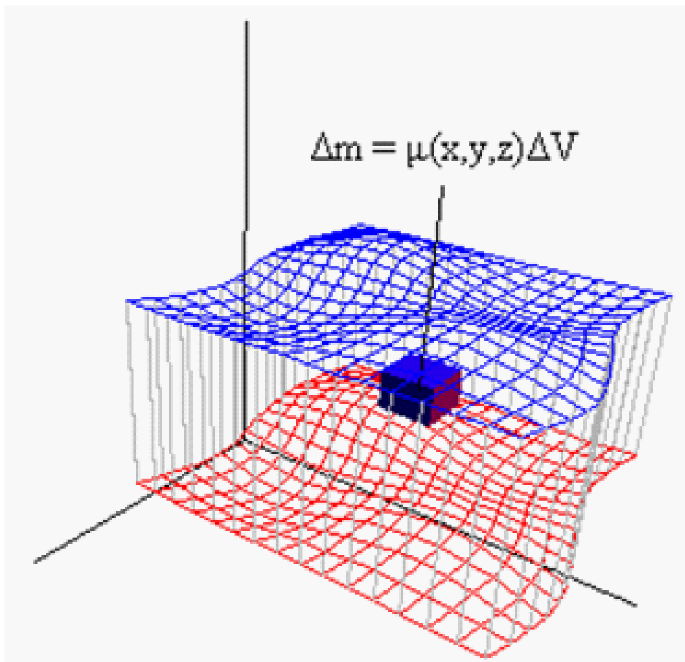
$$M = \iiint_\Omega dM = \iiint_\Omega \mu(x, y, z)\ dV$$

This integral can then be reduced to a triple iterated integral

$$M = \int_0^1 \int_0^1 \int_0^2 \left(9 - z^3\right) dz\ dydx = \int_0^1 \int_0^1 9z - \left.\frac{z^4}{4}\right|_0^2\ dydx$$

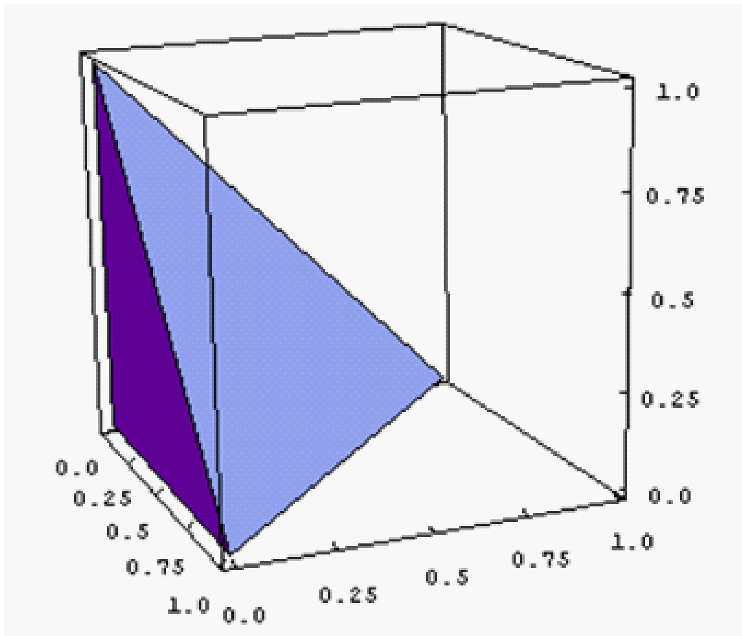
which reduces to $M = 14$ kg, which is about 31 lbs at sea level.

The settling means that near the bottom of the box (where $z \approx 0$), the mass density is about 9 kg per m^3, whereas near the top ($z \approx 2$) the density is about $\mu(x, y, 2) = 9 - 8 = 1$ kg per m^3. Physically, this will means that the *center of mass* should be lower than the point $(0.5, 0.5, 1)$ we would have expected for a uniform mixture, as we will explore in part 4. The key is that the density $\mu(x, y, z)$ is a tool for relating the physical quantity of mass to the mathematical quantity of volume.



Alternatively, densities allow us to imagine that a geometric structure has a physical manifestation.

blue*EXAMPLE 4* blackWhat is the mass of a tetrahedron with vertices at $(0,0,0)$, $(1,0,0,)$, $(0,1,0)$, and $(0,0,1)$ if it has a uniform mass density of $\mu(x, y, z) = 18$ kg per cubic meter.

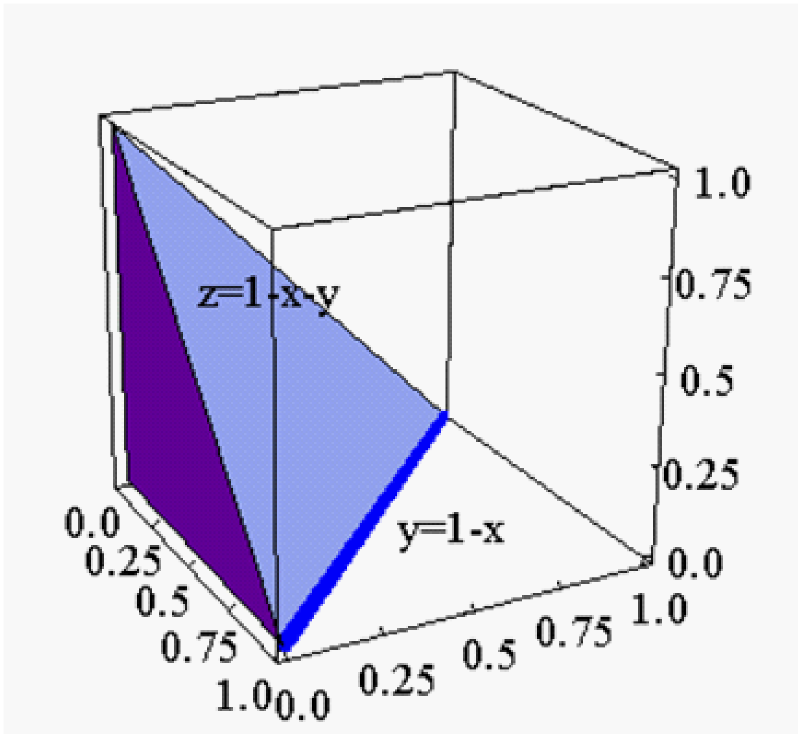


Solution: We begin by identifying the two surfaces that bound the tetrahedron above and below. Since the upper face is flat, it corresponds to the plane through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Since

$$\mathbf{u} = \langle 0 - 1, 1 - 0, 0 - 0 \rangle = \langle -1, 1, 0 \rangle \quad \textit{and} \quad \mathbf{v} = \langle 0 - 1, 0 - 0, 1 - 0 \rangle = \langle -1, 0, 1 \rangle$$

are parallel to the plane, their cross-product $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 1, 1, 1 \rangle$ is normal to the plane, thus implying that the equation of the plane is $z = 1 - x - y$. Thus, the tetrahedron is bounded above by the surface $z = 1 - x - y$ and bounded below by $z = 0$ over the region

R between $x = 0$, $x = 1$, $y = 0$, and $y = 1 - x$ in the xy-plane.



As a result, the mass of the tetrahedron T is

$$M = \iiint_{Tetrahedron} \mu(x, y, z)\, dA = \iint_R \int_0^{1-x-y} 18 dz\ dA$$

Evaluating the first integral yields

$$M = \iint_R 18(1 - x - y)\, dA = \int_0^1 \int_0^{1-x} 18(1 - x - y)\, dy dx = 3$$

Thus, the tetradron has a mass of 3 kg.

Check Your Reading: Geometrically, what are the faces of the tetrahedron?

Other Types of Densities

In general, a *density function* is a function which measures units of a certain quantity per unit volume. For example, in electrostatics we often consider *charge* *densities*, where a charge density $\rho(x, y, z)$ is the amount of charge per unit volume near a point (x, y, z) in space. It follows that the amount of charge dQ in a small region of space with volume dV is given by

$$dQ = \rho(x, y, z)\, dV$$

and as a result, if Ω is a solid which completely contains a "charge cloud," then the total charge Q in Ω is given by

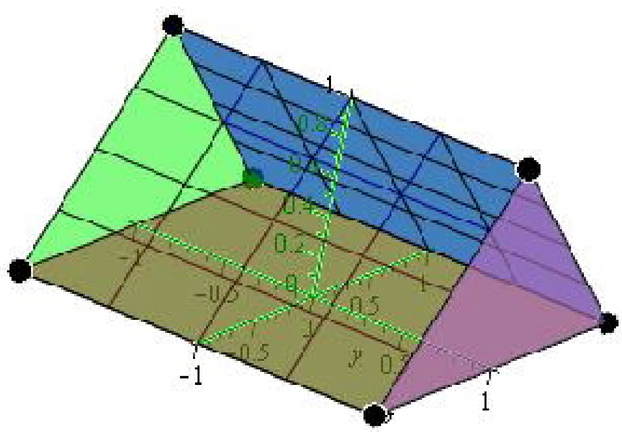
$$Q = \iiint_\Omega dQ = \iiint_\Omega \rho(x, y, z)\, dV$$

Typically, Q is measured in *Coulombs*, so that $\rho(x, y, z)$ is in units of Coulombs per cubic meter.

EXAMPLE 5 What is the charge density of the electron cloud within the prism with vertices at $(1, 1, 0)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 0, 1)$, and $(1, 0, 1)$ given a charge density of

$$\rho(x, y, z) = z\ \frac{C}{m^3}$$

Solution: The pyramid is the solid between the plane P_1 through $(1, -1, 0), (-1, -1, 0), (-1, 0, 1)$, and $(1, 0, 1)$ and the plane P_2 through $(1, 1, 0), (-1, 1, 0), (-1, 0, 1)$, and $(1, 0, 1)$ for x in $[-1, 1]$ and z in $[0, 1]$:



The equation of P_1 is $z - y = 1$, and the equation of P_2 is $z + y = 1$. Since we are integrating over the region $R = [-1, 1] \times [0, 1]$ in the xz-plane, we solve for y in the equations of the planes to obtain

$$P_1 : y = z - 1, \qquad P_2 : y = 1 - z$$

Thus, we must evaluate

$$Q = \iiint_\Omega z \, dV = \iint_R \int_{z-1}^{1-z} z dy \;\; dA_{xz}$$

where $R = [-1, 1] \times [0, 1]$ in the xz-plane. Thus,

$$\begin{aligned} Q &= \int_{-1}^{1} \int_{0}^{1} \quad zy|_{z-1}^{1-z} \, dzdx \\ &= \int_{-1}^{1} \int_{0}^{1} \quad z(2-2z) \;\; dzdx \\ &= \int_{-1}^{1} \int_{0}^{1} \quad \left(2z - 2z^2\right) \;\; dzdx \end{aligned}$$

Consequently, we have

$$Q = \int_{-1}^{1} \quad z^2 - \frac{2z^3}{3}\bigg|_0^1 dx = 2\left(1 - \frac{2}{3}\right) = \frac{2}{3} \; C$$

Other densities can be derived from mass and charge densities. For example, the potential energy U due to of the force of gravitational attraction between two point masses with mass M and m, respectively, is given by

$$U = -G\frac{Mm}{r}$$

where r is the distance between the two points and G is the universal gravitational constant. Thus, a small section of a solid S with mass dM has a potential energy of

$$dU = -G\frac{m}{r} \; dM$$

on an object with mass m which is at a distance r from the small section.

Thus, if the solid has a mass density $\mu(x, y, z)$, then the potential energy of a small section containing the point (x, y, z) is approximately

$$dU = -G\frac{m}{r}\mu(x, y, z) \, dV$$

and the total gravitational potential energy of the solid is

$$U = \iiint_\Omega d\Phi = -Gm \iiint_\Omega \frac{\mu dV}{r}$$

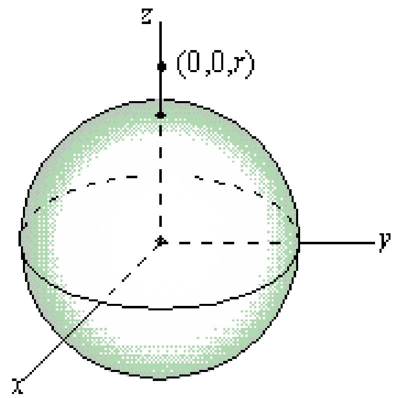
Finally, if the mass m is located at the point (a, b, c), then the distance between the point masses is

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

Thus, the total gravitational potential due to a solid S with a mass-density of $\mu(x,y,z)$ is given by

$$U = -Gm \iiint_{\Omega} \frac{\mu(x,y,z)\,dV}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

EXAMPLE 6 Set up - but do not evaluate – the potential energy of a mass m located at the point $(0,0,r)$ due to the gravitational attraction of a sphere of radius R centered at the origin with a constant mass density.



Solution: Assuming μ is constant and substituting the location $(0,0,r)$ leads to

$$U = -Gm \iiint_{\Omega} \frac{\mu dV}{\sqrt{x^2 + y^2 + (z-r)^2}}$$

where S is the sphere with equation $x^2 + y^2 + z^2 = R^2$.

Check your Reading: How is r related to R in example 6?

Moments and Centers of Mass

If a solid Ω has a mass density of $\mu(x,y,z)$, then its *first moments* are defined to be

$$M_{yz} = \iiint_S x\mu(x,y,z)\,dV, \quad M_{xz} = \iiint_S y\mu(x,y,z)\,dV$$

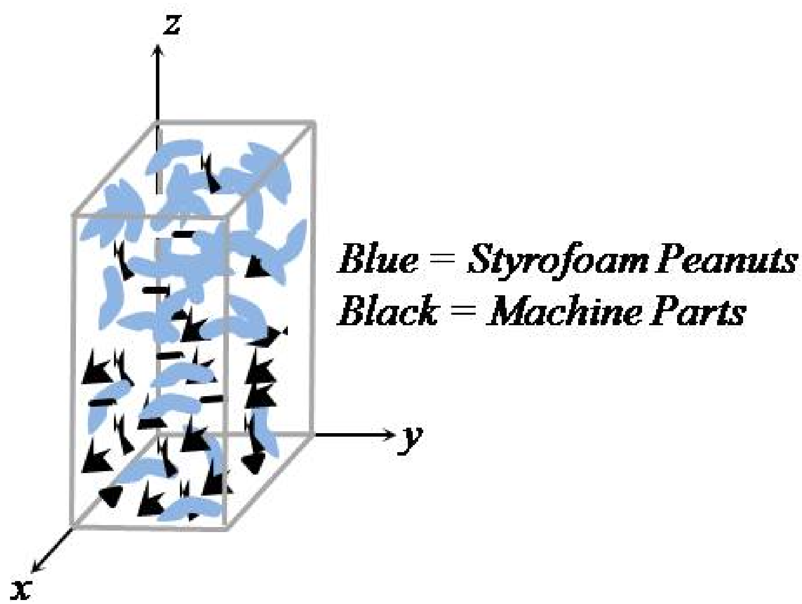
$$M_{xy} = \iiint_S z\mu(x,y,z)\,dV$$

In analogy with the center of mass of a lamina, the *center of mass* of a solid with mass density $\mu(x, y, z)$ is defined to be the point in R^3 with coordinates

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Indeed, these reduce to that of a lamina if $\mu(x, y)$ is uniform in z and the lamina is one unit thick (see the exercises).

EXAMPLE 4 Find the center of mass of the mixture in example 3 – that is, of the peanuts/machine parts mixture in the box $[0, 1] \times [0, 1] \times [0, 2]$



with a mass density of

$$\mu(x, y, z) = \left(9 - z^3\right) \frac{kg}{m^3}$$

Solution: The moments of the box in example 3 are given by

$$\begin{aligned}
M_{yz} &= \iiint_\Omega x\, \mu(x, y, z)\, dV = \int_0^1 \int_0^1 \int_0^2 \left(9 - z^3\right) x\, dzdydx \\
M_{xz} &= \iiint_\Omega y\, \mu(x, y, z)\, dV = \int_0^1 \int_0^1 \int_0^2 \left(9 - z^3\right) y\, dzdydx \\
M_{xy} &= \iiint_\Omega z\, \mu(x, y, z)\, dV = \int_0^1 \int_0^1 \int_0^2 \left(9 - z^3\right) z\, dzdydx
\end{aligned}$$

Evaluating these integrals and computing the coordinates of the center of mass yields

$$\begin{aligned} M_{yz} &= 7\ kg\cdot m, \qquad \bar{x}=\frac{7}{14}=0.5\ m \\ M_{xz} &= 7\ kg\cdot m, \qquad \bar{y}=\frac{7}{14}=0.5\ m \\ M_{xy} &= 11.6\ kg\cdot m, \qquad \bar{z}=\frac{11.6}{14}=0.83\ m \end{aligned}$$

Thus, the center of mass is $(0.5, 0.5, 0.83)$.

That is, the settling of the heavier parts toward the bottom implies a lower center of mass than would have been expected if the parts and peanuts had remained uniformly mixed.

The first moments of a solid Ω with a mass density $\mu(x, y, z)$ are used to determine the center of mass of the solid. Higher moments – moments with nonlinear expressions in x, y, and z are also important. For example, the *moments of inertia* about the three coordinate axes are

$$\begin{aligned} I_x &= \iiint_\Omega \left(y^2+z^2\right)\mu(x,y,z)\,dV, \qquad I_y=\iiint_\Omega \left(x^2+z^2\right)\mu(x,y,z)\,dV \\ I_z &= \iiint_\Omega \left(x^2+y^2\right)\mu(x,y,z)\,dV \end{aligned}$$

Moments of inertia are the rotational analogs of mass. For example, the angular momentum of an object rotated about the z-axis is $L = I_z\omega$, where ω is angular velocity.

blue*EXAMPLE 8* blackWhat are the moments of inertia about the 3 coordinate axes of the machine parts, styrofoam peanuts mixture in example 3, where the mass density is

$$\mu(x,y,z)=\left(9-z^3\right)\ \frac{kg}{m^3}$$

for the solid $[0,1]\times[0,1]\times[0,2]$.

Solution: The moment of inertia about the z-axis is

$$\begin{aligned}
I_z &= \iiint_{[0,1]\times[0,1]\times[0,2]} \left(x^2+y^2\right)\mu\left(x,y,z\right)dV \\
&= \int_0^1\int_0^1\int_0^2 \left(x^2+y^2\right)\left(9-z^3\right)dzdydx \\
&= \int_0^1\int_0^1 \left(x^2+y^2\right)\left(9z-\frac{z^4}{4}\right)\Big|_0^2 dydx \\
&= \int_0^1\int_0^1 17\left(x^2+y^2\right)dydx \\
&= 17\int_0^1 x^2y+\frac{y^3}{3}\Big|_0^1 dx \\
&= 17\int_0^1 \left(x^2+\frac{1}{3}\right)dx \\
&= 17\left(\frac{2}{3}\right) \quad m^2\cdot kg
\end{aligned}$$

Likewise, the moment of inertia about the x-axis is

$$\begin{aligned}
I_x &= \iiint_{[0,1]\times[0,1]\times[0,2]} \left(y^2+z^2\right)\mu\left(x,y,z\right)dV \\
&= \int_0^1\int_0^1\int_0^2 \left(y^2+z^2\right)\left(9-z^3\right)dzdydx \\
&= \int_0^1\int_0^1\int_0^2 \left(9y^2+9z^2-z^5-y^2z^3\right)dzdydx \\
&= \int_0^1\int_0^1 \left(3z^3-\frac{1}{6}z^6+9y^2z-\frac{1}{4}y^2z^4\right)\Big|_0^2 dydx \\
&= \int_0^1\int_0^1 \left(14y^2+\frac{40}{3}\right)dydx \\
&= \int_0^1 \frac{14y^3}{3}+\frac{40y}{3}\Big|_0^1 dx \\
&= 17\int_0^1 \left(\frac{54}{3}\right)dx \\
&= 306 \quad m^2\cdot kg
\end{aligned}$$

By symmetry, $I_x = I_y$.

Exercises:

Find the volume of the solid defined below.

1. $f(x,y) = 1,\ g(x,y) = 0$
   $x = 0,\ x = 1,\ y = 0,\ y = 1$
2. $f(x,y) = x + 2,\ g(x,y) = 0$
   $x = 1,\ x = 2,\ y = 0,\ y = 3$
3. $f(x,y) = xy,\ g(x,y) = 0$
   $y = 0,\ y = 1,\ x = y,\ x = 1$
4. $f(x,y) = x^2 + xy,\ g(x,y) = 0$
   $y = 0,\ y = 1,\ x = y,\ x = y^2$
5. $f(x,y) = x + y,\ g(x,y) = x^2 + y^2$
   $x = 0,\ x = 1,\ y = 0,\ y = 1$
6. $f(x,y) = xy,\ g(x,y) = 4$
   $y = 0,\ y = 1,\ x = y,\ x = 1$

Find the mass of the solid defined below with the given mass density.

7. $f(x,y) = xy,\ g(x,y) = 0$
   $x = 0,\ x = 1,\ y = 0,\ y = 1$
   $\mu(x,y,z) = 2$ kg per cubic meter
8. $f(x,y) = x + 2y,\ g(x,y) = 0$
   $x = 1,\ x = 2,\ y = 0,\ y = 6$
   $\mu(x,y,z) = 2$ kg per cubic meter
9. $f(x,y) = x^2 + y^2,\ g(x,y) = 0$
   $y = 0,\ y = 1,\ x = y,\ x = 1$
   $\mu(x,y,z) = 2x$ kg per cubic meter
10. $f(x,y) = x^3 + y^2,\ g(x,y) = 0$
    $y = 1,\ y = 2,\ x = y,\ x = y^2$
    $\mu(x,y,z) = 2z$ kg per cubic meter
11. $f(x,y) = x + y,\ g(x,y) = x^2 + y^2$
    $x = 0,\ x = 1,\ y = 0,\ y = 1$
    $\mu(x,y,z) = 2y$ kg per cubic meter
12. $f(x,y) = xy,\ g(x,y) = 4$
    $y = 0,\ y = 1,\ x = y,\ x = 1$
    $\mu(x,y,z) = 2z$ kg per cubic meter

Find the total charge within the solid defined below with the given charge density (m = meter).

13. $f(x,y) = 1,\ g(x,y) = 0$
    $x = 0,\ x = 1,\ y = 0,\ y = 1$
    $\rho(x,y,z) = 2$ coulombs per m^3
14. $f(x,y) = 4,\ g(x,y) = 2$
    $x = 1,\ x = 2,\ y = 1,\ y = 6$
    $\rho(x,y,z) = 5$ coulombs per m^3
15. $f(x,y) = 1,\ g(x,y) = 0$
    $y = 0,\ y = 1,\ x = y,\ x = 1$
    $\rho(x,y,z) = 2z$ coulombs per m^3
16. $f(x,y) = x^3 + y^2,\ g(x,y) = 0$
    $y = 1,\ y = 2,\ x = y,\ x = y^2$
    $\rho(x,y,z) = yz$ coulombs per m^3
17. $f(x,y) = x + y,\ g(x,y) = x^2 + y^2$
    $x = 0,\ x = 1,\ y = 0,\ y = 1$
    $\rho(x,y,z) = x + y$ coulombs per m^3
18. $f(x,y) = xy,\ g(x,y) = 4$
    $y = 0,\ y = 1,\ x = y,\ x = 1$
    $\rho(x,y,z) = 2x$ coulombs per m^3

Find the center of mass of the solid defined below with the given mass density. (see 7-12 for the masses) Also, find the moments of inertia about the 3 coordinate axes.

19. $f(x,y) = xy,\ g(x,y) = 0$
    $x = 0,\ x = 1,\ y = 0,\ y = 1$
    $\mu(x,y,z) = 2$ kg per cubic meter
20. $f(x,y) = x + 2y,\ g(x,y) = 0$
    $x = 1,\ x = 2,\ y = 0,\ y = 6$
    $\mu(x,y,z) = 2$ kg per cubic meter
21. $f(x,y) = x^2 + y^2,\ g(x,y) = 0$
    $y = 0,\ y = 1,\ x = y,\ x = 1$
    $\mu(x,y,z) = 2x$ kg per cubic meter
22. $f(x,y) = x^3 + y^2,\ g(x,y) = 0$
    $y = 1,\ y = 2,\ x = y,\ x = y^2$
    $\mu(x,y,z) = 2z$ kg per cubic meter
23. $f(x,y) = x + y,\ g(x,y) = x^2 + y^2$
    $x = 0,\ x = 1,\ y = 0,\ y = 1$
    $\mu(x,y,z) = 2y$ kg per cubic meter
24. $f(x,y) = xy,\ g(x,y) = 4$
    $y = 0,\ y = 1,\ x = y,\ x = 1$
    $\mu(x,y,z) = 2z$ kg per cubic meter

25. What is the volume of a right circular cylinder whose height is h and whose base has a radius of r?

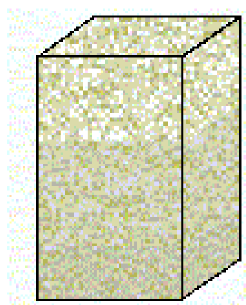
26. What is the volume of a regular pyramid with a height h and a square base with sides of length s?

27. A certain type of concrete has a weight density at the earth's surface of 10 pounds per cubic foot. What is the weight of a concrete block in the shape of the solid

$$x = 0, x = 1, y = 0, y = 1, z = 0, z = y^2$$

where all dimensions are in feet?

28. A mixture of sand and gravel is placed in a box with a square base of width 1 meter and a height of 2 meters. The box is then shaken vigorously causing more of the gravel to be near the bottom and more of the sand to be near the top.



What is the mass of the sand-gravel mixture in the box if it has a mass density of

$$\mu(x, y, z) = (3 - z) \ \textit{kg per } m^3$$

29. What is the center of mass of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ if it has a uniform mass density of 36 kg per cubic meter?

30. What is the mass of the pyramid with vertices $(1,0,1)$, $(1,0,-1)$, $(-1,0,1)$, $(-1,0,-1)$, and $(0,2,0)$ if the mass density is $\rho(x,y,z) = 3 - z$ kg per cubic meter?

31. A *current density* is a density of the form

$$\mathbf{j}(x, y, z) = \rho_m(x, y, z) \ \mathbf{v}(x, y, z)$$

where $\rho_m(x,y,z)$ is the density of the moving charges within a small section of a solid containing (x,y,z) and $\mathbf{v}(x,y,z)$ is the velocity vector of a charge at (x,y,z) if it is moving. What is the triple integral which represents the *total current* within the right circular cylinder

$$x^2 + z^2 = R^2$$

where R is a constant and where y is in $[0, h]$?

32. Suppose that a solid S with mass dM produces a potential energy of

$$dU = -kmr \, dM$$

on an object with mass m which is at a distance r from the small section. What is the triple integral form of the total potential energy when the solid has a mass density of $\mu(x,y,z)$?

Exercises 33-39 deal with delta densities and point masses. Earlier exercises must be precede later exercises in this set.

33. The *delta density* $\delta(x,y,z)$ is a density which satisfies

$$\iiint_S \delta(x,y,z)\,dV = \begin{cases} 1 & if \quad (0,0,0) \text{ is in } S \\ 0 & if \quad (0,0,0) \text{ is not in } S \end{cases}$$

For example, the mass density of a solid S in which all of the mass M is "concentrated" at the origin is given by

$$\mu(x,y,z) = M\ \delta(x,y,z)$$

What is the total mass of a solid S with such a mass density?

34. Suppose that all of the mass M of a solid S is "concentrated" at the point (a,b,c). Explain why the mass density for S is

$$\mu(x,y,z) = M\ \delta(x-a, y-b, z-c)$$

What is the total mass of a solid S with such a mass density?

35. Suppose a solid S has a charge density of

$$\rho(x,y,z) = q_1\delta(x,y,z) + q_2\delta(x-a, y-b, z-c)$$

What does this density tell us about the charges inside of S? What is the total charge within S?

36. What is the center of mass of a solid S with a mass density of

$$\mu(x,y,z) = M_1\delta(x,y,z) + M_2\delta(x-a, y-b, z-c)$$

37. Write to Learn: In vector notation, we let $\mathbf{r} = \langle x,y,z\rangle$. Thus, if $\mathbf{r}_0 = \langle a,b,c\rangle$, then

$$\delta(x-a, y-b, z-c) = \delta(\mathbf{r}-\mathbf{r}_0)$$

Write a short essay explaining why a collection of point masses $m_1,\ldots,m_n$ located at points

$$\mathbf{r}_1,\ldots,\mathbf{r}_n$$

has a mass density of

$$\mu(\mathbf{r}) = \sum_{j=1}^{n} m_j\delta(\mathbf{r}-\mathbf{r}_j)$$

and then find the total mass of the collection. **Bonus:** What is the center of mass of the collection?

38. Write to Learn: The delta density $\delta(x,y,z)$ is more often defined by

$$\iiint_S \delta(x,y,z)\,f(x,y,z)\,dV = \begin{cases} f(0,0,0) & if \quad (0,0,0) \text{ is in } S \\ 0 & if \quad (0,0,0) \text{ is not in } S \end{cases}$$

when $f(x,y,z)$ is a continuous function. Write a short essay in which you show that this definition reduces to the one in exercise 31 and that in addition, we have

$$\iiint_S \delta(x-a, y-b, z-c)\, f(x,y,z)\, dV = f(a,b,c)$$

when (a,b,c) is in S.

39. Show that if we define

$$\int_a^b \delta(x)\, dx = \begin{cases} 1 & if \quad 0 \; is \; in \; [a,b] \\ 0 & if \quad 0 \; is \; not \; in \; [a,b] \end{cases}$$

then $\delta(x,y,z) = \delta(x)\,\delta(y)\,\delta(z)$. (i.e., show that it satisfies the definition in exercise 31).

40. Use Fubini's theorem for double integrals to prove Fubini's theorem for triple integrals, which says that

$$\begin{aligned} \int_a^b \int_c^d \int_e^f \phi(x,y,z)\, dzdydx &= \int_c^d \int_a^b \int_e^f \phi(x,y,z)\, dzdxdy \\ &= \int_c^d \int_e^f \int_a^b \phi(x,y,z)\, dxdzdy \\ &= \int_e^f \int_c^d \int_a^b \phi(x,y,z)\, dxdydz \end{aligned}$$