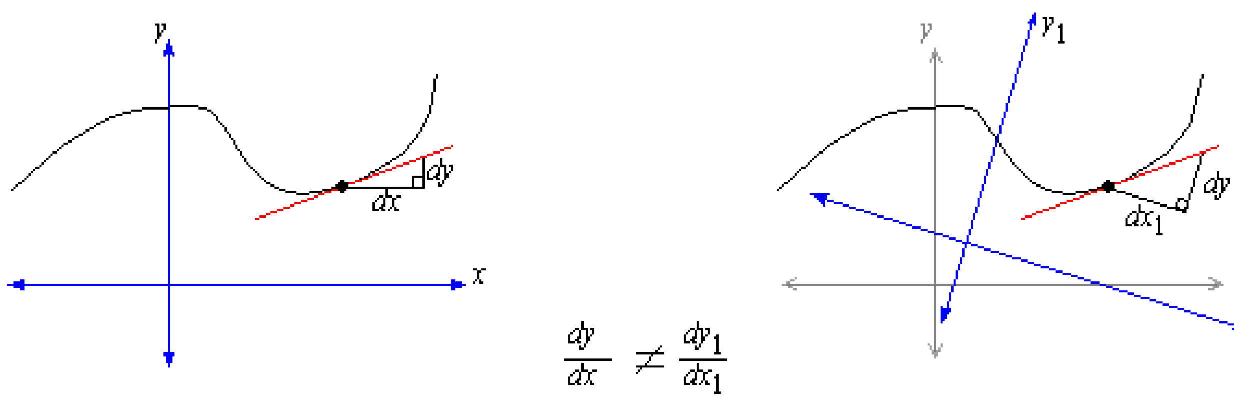
Intrinsic Geometry

The Fundamental Form of a Surface

Properties of a curve or surface which depend on the coordinate space that curve or surface is embedded in are called extrinsic properties of the curve. For example, the slope of a tangent line is an extrinsic property since it depends on the coordinate system in which rises and runs are measured.



In contrast, intrinsic properties of surfaces are properties that can be measured within the surface itself without any reference to a larger space.

For example, the length of a curve is an intrinsic property of the curve, and thus, the length of a curve $\boldsymbol{\rho}(t) = \mathbf{r}(u(t), v(t))$, t in $[a, b]$, on a surface $\mathbf{r}(u, v)$ is an intrinsic property of both the curve itself and the surface that contains it. As we saw in the last section, the square of the speed of $\boldsymbol{\rho}(t)$ is

$$\left(\frac{ds}{dt}\right)^2 = g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + g_{22}\left(\frac{dv}{dt}\right)^2$$

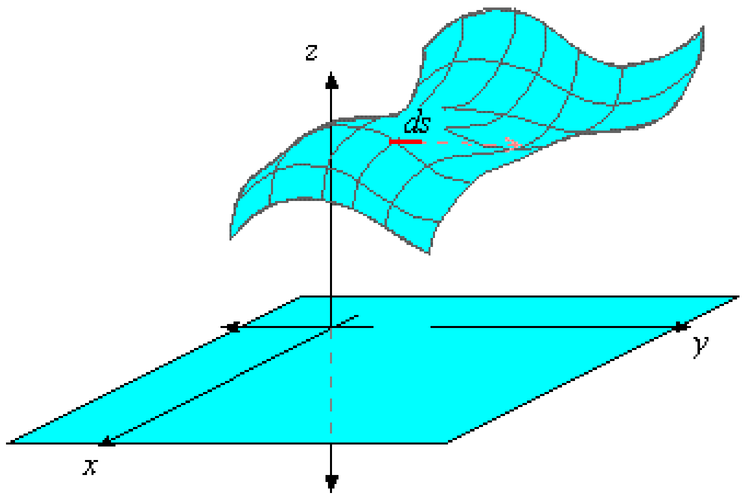
in terms of the *metric coefficients*

$$g_{11} = \mathbf{r}_u \cdot \mathbf{r}_u, \quad g_{12} = \mathbf{r}_v \cdot \mathbf{r}_u, \quad \textit{and} \quad g_{22} = \mathbf{r}_v \cdot \mathbf{r}_v$$

Thus, very short distances ds on the surface can be approximated by

$$(ds)^2 = g_{11}(du)^2 + 2g_{12}du dv + g_{22}(dv)^2 \tag{1}$$

That is, if du and dv are sufficiently small, then ds is the length of an infinitesimally short curve on the surface itself.



Equation (1) is the *fundamental form* of the surface, which intrinsic to a surface because it is related to distances on the surface itself. Moreover, any properties which can be derived *solely from a surface's fundamental form* are also intrinsic to the surface.

> *EXAMPLE 1* Find the fundamental form of the right circular cylinder of radius R, which can be parameterized by
>
> $$\mathbf{r}(u,v) = \langle R\cos(u), R\sin(u), v\rangle$$
>
> **Solution:** Since $\mathbf{r}_u = \langle -R\sin(u), R\cos(u), 0\rangle$ and $\mathbf{r}_v = \langle 0,0,1\rangle$, the metric coefficients are
>
> $$\begin{aligned} g_{11} &= \mathbf{r}_u \cdot \mathbf{r}_u = R^2\sin^2(u) + R^2\cos^2(u) + 0^2 = R^2 \\ g_{12} &= \mathbf{r}_u \cdot \mathbf{r}_v = 0+0+0 = 0 \\ g_{22} &= \mathbf{r}_v \cdot \mathbf{r}_v = 0^2+0^2+1^2 = 1 \end{aligned}$$
>
> Thus, $ds^2 = R^2du^2 + dv^2$.

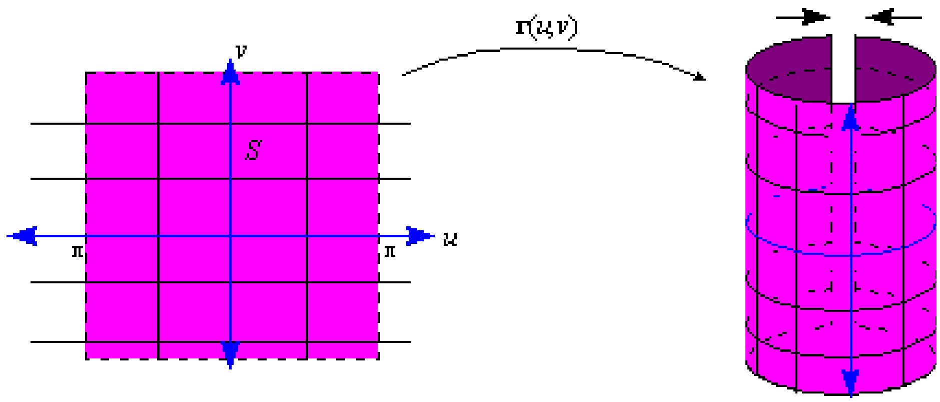
If the parameterization is orthogonal, then $g_{12} = \mathbf{r}_u \cdot \mathbf{r}_v = 0$, so that

$$ds^2 = g_{11}du^2 + g_{22}dv^2$$

For example, the xy-plane is parameterized by $r(u,v) = \langle u, v, 0\rangle$,which implies that $\mathbf{r}_u = \mathbf{i}$ and $\mathbf{r}_v = \mathbf{j}$ and that $g_{11} = g_{22} = 1$, $g_{12} = 0$. The fundamental form for the plane is

$$ds^2 = du^2 + dv^2$$

which is, in fact, the Pythagorean theorem. Moreover, distances are not altered when a "sheet of paper" is rolled up into a cylinder, which means that a cylinder should have the same fundamental form as the plane.



Indeed, if $R = 1$ in example 1, then $ds^2 = du^2 + dv^2$.

EXAMPLE 2 Find the fundamental form of the sphere of radius R centered at the origin in the spherical coordinate parametrization

$$\mathbf{r}(\phi, \theta) = \langle R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)\rangle$$

Solution: To do so, we first compute the derivatives $\mathbf{r}_\phi$ and $\mathbf{r}_\theta$:

$$\begin{aligned}
\mathbf{r}_\phi &= \langle R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)\rangle \\
\mathbf{r}_\theta &= \langle -R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0\rangle
\end{aligned}$$

It then follows that

$$\begin{aligned}
\mathbf{r}_\phi \cdot \mathbf{r}_\phi &= R^2\cos^2(\phi)\cos^2(\theta) + R^2\cos^2(\phi)\sin^2(\theta) + R^2\sin^2(\phi) \\
&= R^2\cos^2(\phi) + R^2\sin^2(\phi) \\
&= R^2
\end{aligned}$$

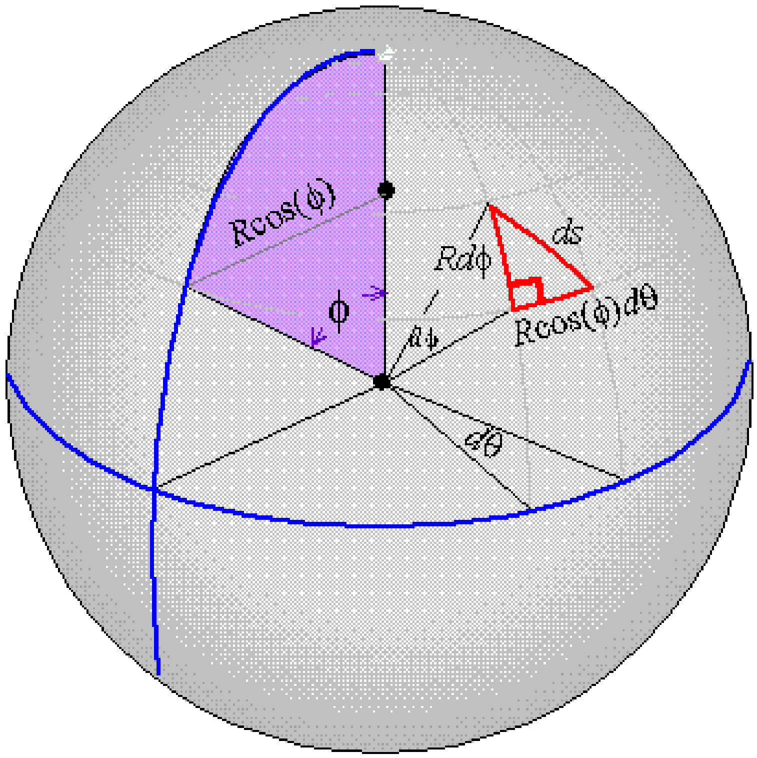
and thus, $g_{11} = R^2$. Moreover, spherical coordinates is an orthogonal parameterization, which means that $\mathbf{r}_\phi \cdot \mathbf{r}_\theta = 0$. Thus, $g_{12} = 0$. Finally, g_{22} is given by

$$g_{22} = \mathbf{r}_\theta \cdot \mathbf{r}_\theta = R^2\sin^2(\phi)\sin^2(\theta) + R^2\sin^2(\phi)\cos^2(\theta) = R^2\sin^2(\phi)$$

As a result, the fundamental form of the sphere of radius R is given by

$$ds^2 = R^2 d\phi^2 + R^2\sin^2(\phi)\, d\theta^2 \tag{2}$$

That is, the "hypotenuse" of a spherical "right triangle" corresponds to a "horizontal" arc of length $R\sin(\phi)\,d\theta$ and a "vertical" arc of length $Rd\phi$.



Check Your Reading: Is the Pythagorean theorem intrinsic to the xy-plane?

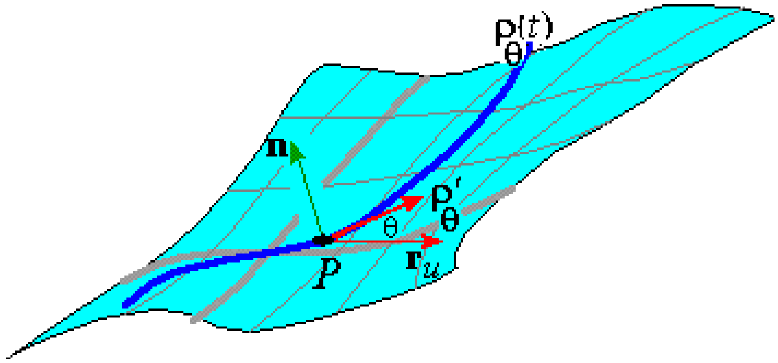
Normal Curvature

If P is a point on an orientable surface Σ and if $\mathbf{r}(u,v)$ is an orthogonal parameterization of a coordinate patch on Σ containing P, then there is (p,q) such that $\mathbf{r}(p,q)=P$. Thus, for each θ in $[0,2\pi]$, the curves

$$\boldsymbol{\rho}_\theta(t)=\mathbf{r}(p+t\cos(\theta),q+t\sin(\theta))$$

pass through P (i.e., $\boldsymbol{\rho}_\theta(0)=\mathbf{r}(p,q)=P$) and the tangent vectors $\boldsymbol{\rho}'_\theta(0)$

point in a different direction in the tangent plane to Σ at P.



Indeed, every direction in the tangent plane is parallel to $\boldsymbol{\rho}'_\theta(0)$ for some θ.

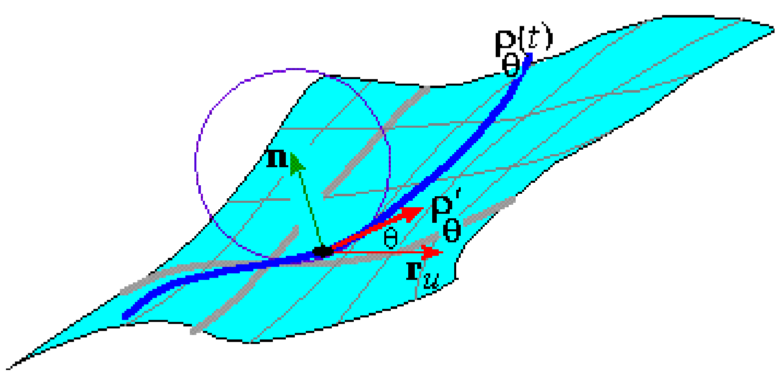
Consequently, the curvatures of $\boldsymbol{\rho}_\theta(t)$ at P represent all the different ways that the surface "curves away" from the tangent plane at P. Correspondingly, we define $\kappa_{\mathbf{n}}(\theta)$ to be the component of the curvature of $\boldsymbol{\rho}_\theta(t)$ at P in the direction of the unit normal

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

In particular, the normal curvature satisfies $v^2\ \kappa_{\mathbf{n}}(\theta) = \boldsymbol{\rho}''_\theta(0) \cdot \mathbf{n}$, so that

$$\kappa_{\mathbf{n}}(\theta) = \frac{\boldsymbol{\rho}''_\theta(0) \cdot \mathbf{n}}{\|\boldsymbol{\rho}'_\theta(0)\|^2} \tag{3}$$

The unit surface normal $\mathbf{n}$ is not necessarily the same as the unit normal $\mathbf{N}$ for the curve. They may point in opposite directions or even be orthogonal – e.g., the normal $\mathbf{N}$ to a curve in the xy-plane is in the xy-plane itself yet $\mathbf{n} = \mathbf{k}$ for the xy-plane. Thus, the normal curvature $\kappa_{\mathbf{n}}(\theta)$ is a measure of how much the surface is curving rather than how much the curve is curving.



That means that $\boldsymbol{\kappa}_n(\theta)$ may be positive, negative, or even 0 – e.g., the normal curvature of the xy-plane is 0 even though there are curves in the xy-plane with nonzero curvature.

EXAMPLE 3 Use (**??**) to find the normal curvature $\kappa_{\mathbf{n}}(\theta)$ for the cylinder

$$\mathbf{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$$

at the point $\mathbf{r}(0,0) = (1,0,0)$.

Solution: The curves $\boldsymbol{\rho}_{\theta}(t) = \mathbf{r}(t\cos(\theta), t\sin(\theta))$ are given by

$$\boldsymbol{\rho}_{\theta}(t) = \langle \cos(t\cos(\theta)), \sin(t\cos(\theta)), t\sin(\theta) \rangle$$

Differentiation with respect to t leads to

$$\begin{aligned} \boldsymbol{\rho}_{\theta}' &= \langle -\sin(t\cos(\theta))\cos(\theta), \cos(t\cos(\theta))\cos(\theta), \sin(\theta) \rangle \\ \boldsymbol{\rho}_{\theta}'' &= \left\langle -\cos(t\cos(\theta))\cos^2(\theta), -\sin(t\cos(\theta))\cos^2(\theta), 0 \right\rangle \end{aligned}$$

so that at $t = 0$ we have

$$\begin{aligned} \boldsymbol{\rho}_{\theta}'(0) &= \langle -\sin(0)\cos(\theta), \cos(0)\cos(\theta), \sin(\theta) \rangle = \langle 0, \cos(\theta), \sin(\theta) \rangle \\ \boldsymbol{\rho}_{\theta}''(0) &= \left\langle -\cos(0)\cos^2(\theta), -\sin(0)\cos^2(\theta), 0 \right\rangle = \left\langle -\cos^2(\theta), 0, 0 \right\rangle \end{aligned}$$

The partial derivatives of $\mathbf{r}(u,v)$ are

$$\mathbf{r}_u = \langle -\sin(u), \cos(u), 0 \rangle, \qquad \mathbf{r}_v = \langle 0, 0, 1 \rangle$$

which are both unit vectors. It follows that $\mathbf{r}_u(0,0) = \langle 0,1,0 \rangle = \mathbf{j}$ and $\mathbf{r}_v = \langle 0,1,0 \rangle = \mathbf{k}$. Thus,

$$\mathbf{n} = \mathbf{r}_u(0,0) \times \mathbf{r}_v(0,0) = \mathbf{j} \times \mathbf{k} = \mathbf{i}$$

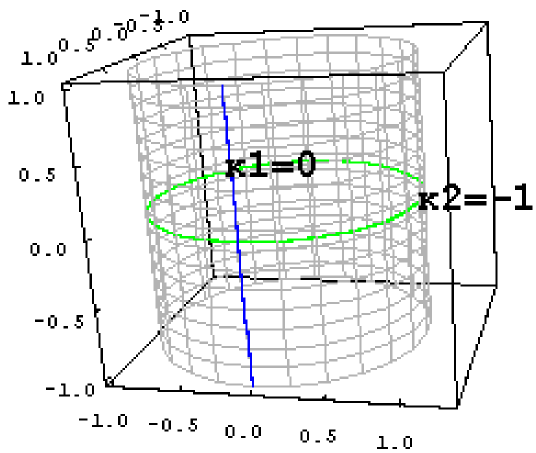
Since $\rho_{\theta}'(0)$ is a unit vector, we have

$$\kappa_{\mathbf{n}}(\theta) = \frac{\boldsymbol{\rho}_{\theta}''(0) \cdot \mathbf{n}}{\|\boldsymbol{\rho}_{\theta}'(0)\|^2} = \frac{\left\langle -\cos^2(\theta), 0, 0 \right\rangle \cdot \mathbf{i}}{1} = -\cos^2(\theta)$$

Because the normal curvature $\kappa_{\mathbf{n}}(\theta)$ is a real-valued continuous function over θ in $[0, 2\pi]$, there is a largest κ_1 and a smallest κ_2 curvature at each point. The numbers κ_1 and κ_2 are known as the *principle curvatures* of the surface $\mathbf{r}(u,v)$.

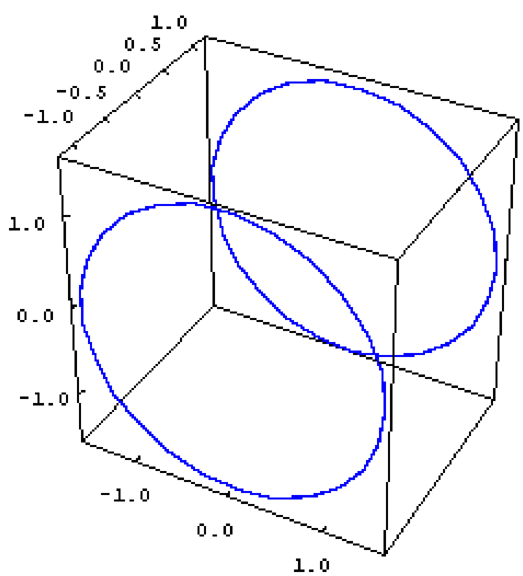
In example 3, the largest possible curvature is $\kappa_1 = -\cos^2(\pi/2) = 0$ in the vertical direction, and the smallest possible curvature is $\kappa_2 = -\cos^2(0) = -1$

in the horizontal direction.



The *average* of the principal curvatures is called the *Mean curvature* of the surface and is denoted by H. It can be shown that if C is a sufficiently smooth closed curve, then the surface with C as its boundary curve that has the smallest possible area must have a mean curvature of $H = 0$.

For example, consider two parallel circles in 3 dimensional space.



The *catenoid* is the surface connecting the two circles that has the *least surface area*, and a catenoid also has a mean curvature of $H = 0$.

Surfaces with a mean curvature of H = 0 are called minimal surfaces because they also have the least surface area for their given boundary. For example, a soap film spanning a wire loop is a minimal surface. Moreover, minimal surfaces have a large number of applications in architecture, mathematics, and engineering.

Check your Reading: Is the normal curvature of the cylinder ever positive?

Gaussian Curvature

In contrast to the mean curvature of a surface, the product of the principal curvatures is known as the *Gaussian curvature* of the surface, which is denoted by K. For example, the Gaussian curvature of the cylinder in example 2 is $K = -1 \cdot 0 = 0$ and the Gaussian curvature of the Enneper minimal surface is

$$K = \frac{-2}{(u^2+v^2+1)^2} \cdot \frac{2}{(u^2+v^2+1)^2} = \frac{-4}{(u^2+v^2+1)^4}$$

As another example, consider that since the geodesics of a sphere with radius R are great circles of the sphere, they each have a curvature of $\kappa = 1/R$. Thus, the principal curvatures must be $\kappa_1 = 1/R$ and $\kappa_2 = 1/R$, so that the curvature of a sphere of radius R is

$$K = \frac{1}{R^2}$$

Curvature is in general an extrinsic property of a surface. For example, mean curvature H is extrinsic because it depends on how the surface is embedded in a 3 (or higher) dimensional coordinate system. In contrast, the Gaussian curvature K is an *intrinsic property* of the surface. This is a truly remarkable theorem, one that was proven by the mathematician Karl Gauss in 1827.

In particular, if we use the *comma derivative* notation to denote partial derivatives of metric coefficients with respect to u and v

$$g_{11,u} = \frac{\partial g_{11}}{\partial u} \quad \text{and} \quad g_{22,v} = \frac{\partial g_{22}}{\partial v}$$

then we can state Gauss' remarkable theorem as follows:

> **Theorem Egregium:** Let $\mathbf{r}(u,v)$ be an orthogonal parameterization of a surface and let $g = (g_{11}g_{22})^{1/2}$ be the *metric discrimanent* of the fundamental form of $\mathbf{r}(u,v)$. Then K is an instrinsic property of the surface since can be expressed in terms of the metric

coefficients by

$$K = \frac{-1}{2g}\left[\frac{\partial}{\partial v}\left(\frac{g_{11,v}}{g}\right) + \frac{\partial}{\partial u}\left(\frac{g_{22,u}}{g}\right)\right] \tag{4}$$

Since K is given in terms of the metric coefficients of the fundamental form, it is an intrinsic property of the surface.

blue*EXAMPLE 4* blackIn example 2, we showed that the fundamental form of a sphere of radius R is given by

$$ds^2 = R^2 d\phi^2 + R^2 \sin^2(\phi)\, d\theta^2$$

Use the Theorem Egregium to calculate the curvature of the sphere.

Solution: Since $g_{11} = R^2$ and $g_{22} = R^2 \sin^2(\phi)$, the comma derivatives are

$$g_{11,\theta} = 0 \qquad and \qquad g_{22,\phi} = 2R^2 \sin(\phi)\cos(\phi)$$

The metric discriminant is given by

$$g = \sqrt{R^2\left(R^2 \sin^2 \phi\right)} = R^2 \sin(\phi)$$

so that

$$\begin{aligned} K &= \frac{-1}{2R^2\sin(\phi)}\left[\frac{\partial}{\partial\theta}\left(\frac{0}{R^2\sin(\phi)}\right) + \frac{\partial}{\partial\phi}\left(\frac{2R^2\sin\phi\cos(\phi)}{R^2\sin(\phi)}\right)\right] \\ &= \frac{-1}{2R^2\sin(\phi)}\left[0 + \frac{\partial}{\partial\phi}(2\cos(\phi))\right] \\ &= \frac{-1}{2R^2\sin(\phi)}\left[-2\sin(\phi)\right] \\ &= \frac{1}{R^2} \end{aligned}$$

The Theorem Egregium is simpler still – and also more easily interpreted – for a parameterization $\mathbf{r}(u,v)$ that is *conformal.* Specifically, a conformal parameterization is orthogonal and has $\|\mathbf{r}_u\| = \|\mathbf{r}_v\|$, which in turn implies that

$$g = g_{11} = g_{22}$$

The formula for K reduces in this case to

$$\begin{aligned} K &= \frac{-1}{2g}\left[\frac{\partial}{\partial v}\left(\frac{g_v}{g}\right)+\frac{\partial}{\partial u}\left(\frac{g_u}{g}\right)\right] \\ &= \frac{-1}{2g}\left[\frac{\partial}{\partial v}\left(\frac{\partial}{\partial v}\ln(g)\right)+\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\ln(g)\right)\right] \end{aligned}$$

since $\partial_v \ln(g) = g_v/g$ and $\partial_u \ln(g) = g_u/g$. That is,

$$K = \frac{-1}{2g}\left[\frac{\partial^2 \ln(g)}{\partial u^2}+\frac{\partial^2 \ln(g)}{\partial v^2}\right] \tag{5}$$

for a conformal surface.

EXAMPLE 5 A coordinate patch on a right circular cone is parameterized by

$$\mathbf{r}(u,v) = \left\langle e^u \cos\left(\sqrt{2}v\right), e^u \sin\left(\sqrt{2}v\right), e^u\right\rangle$$

Find its fundamental form and use the Theorem Egregium to calculate the curvature of the surface.

Solution: Since $\mathbf{r}_u = \left\langle e^u \cos\left(\sqrt{2}v\right), e^u \sin\left(\sqrt{2}v\right), e^u\right\rangle$ and

$$\mathbf{r}_v = \left\langle -e^u\sqrt{2}\sin\left(\sqrt{2}v\right), e^u\sqrt{2}\cos\left(\sqrt{2}v\right), 0\right\rangle$$

the metric coefficients are

$$\begin{aligned} g_{11} &= \mathbf{r}_u\cdot\mathbf{r}_u = e^{2u}\cos^2(v)+e^{2u}\sin^2(v)+e^{2u} = 2e^{2u} \\ g_{22} &= \mathbf{r}_v\cdot\mathbf{r}_v = 2e^{2u}\sin^2(v)+2e^{2u}\cos^2(v) = 2e^{2u} \end{aligned}$$

Thus, the fundamental form is conformal

$$ds^2 = 2e^{2u}\left(du^2+dv^2\right)$$

and $g = e^{2u}$. Thus,

$$\begin{aligned} K &= \frac{-1}{2e^{2u}}\left[\frac{\partial^2 \ln\left(e^{2u}\right)}{\partial u^2}+\frac{\partial^2 \ln\left(e^{2u}\right)}{\partial v^2}\right] \\ &= \frac{-1}{2e^{2u}}\left[\frac{\partial^2(2u)}{\partial u^2}+\frac{\partial^2(2u)}{\partial v^2}\right] \\ &= 0 \end{aligned}$$

Because K is intrinsic, the Gaussian curvature of a surface can be measured by "inhabitants" of the surface without regard to the larger space the surface is embedded in. For example, a person could measure the curvature of the earth's surface even if he was blind and could not see that the earth was situated in a larger cosmos.

Similarly, if one surface is mapped to another without changing distances, then the two surfaces are said to be *isometric.* Intrinsic properties, such as Gaussian curvature, are invariant over isometries. For example, a plane has a Gaussian curvature of 0 and a cylinder likewise has a Gaussian curvature of 0, which follows from the fact that a plane can be rolled up into a cylinder without "stretching" or "tearing" – that is, without changing distances between points.

Check your Reading: What is the Mean curvature of a sphere of radius R?

The Poincare' Half-Plane

Intrinsic geometry also means that we can define and study abstract surfaces that cannot be embedded in 3-dimensional space; or similarly, that we can determine the intrinsic geometric properties of space-time without having "space-time" embedded in a larger space.

For example, we can define a *new* geometry on the plane by giving it a *non-Euclidean* fundamental form. How would we know that it was truly different? This is exactly what Henri' Poincare' did when he introduced the fundamental form

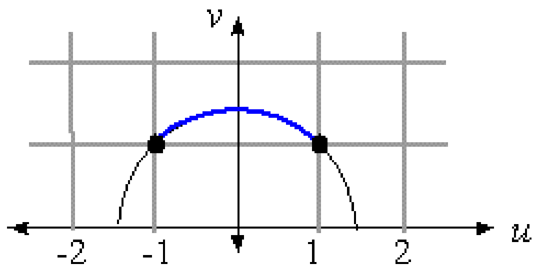
$$ds^2 = \frac{du^2 + dv^2}{v^2} \tag{6}$$

to the upper half of the uv-plane. The result is called the *Poincare half-plane* and is a model of *hyperbolic geometry.*

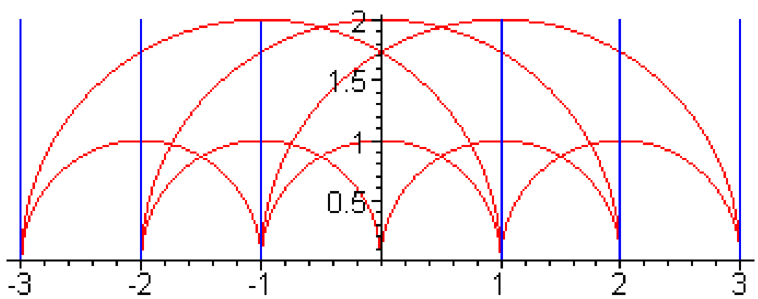
If we use (6) to measure distances, then the geodesics are the vertical lines and semicircles parameterized by

$$u = R\tanh(t) + p, \quad v = R\,\mathrm{sech}(t), \quad t\ in\ (-\infty, \infty)$$

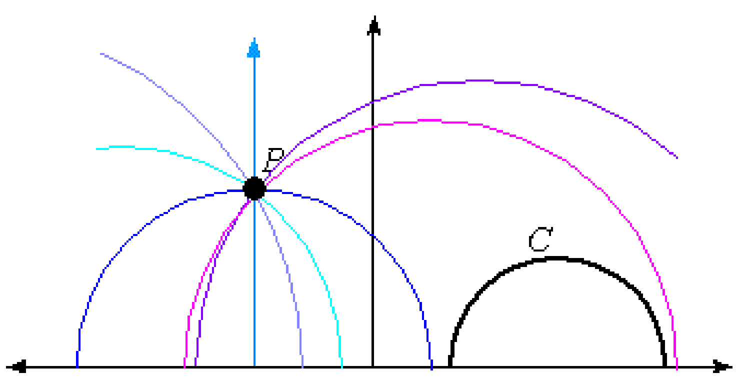
for R and p constant. For example, because distances become shorter as v increases *under the Poincare metric (6),* the distance from $(-1,1)$ to $(1,1)$ along a semi-circle of radius $\sqrt{2}$ centered at the origin is 1.7627, which is shorter than the distance of 2 from $(-1,1)$ to $(1,1)$ along the line $v=1$.



Thus, vertical lines and semi-circles centered on the u-axis are the "straight lines" in the *Poincare half-plane.* Through a point P not on a semi-circle, there are infinitely many other semi-circles centered on the x-axis that pass through P.



Thus, in the *Poincare half plane*, there are infinitely many "parallel lines" to a given "line" l through a point P not on l.



Finally, we can use the Theorem Egregium to calculuate the curvature of the Poincare half-plane. In particular, since $ds^2 = v^{-2}du^2 + v^{-2}dv^2$, the metric coefficients are $g_{11} = g_{22} = v^{-2}$. Thus,

$$g_{11,v} = -2v^{-3}, \quad g_{22,u} = 0, \quad and \quad g = g_{11}g_{22} = v^{-4}$$

The theorem Egregium thus yields

$$\begin{aligned} K &= \frac{-1}{2\sqrt{v^{-4}}}\left[\frac{\partial}{\partial v}\left(\frac{-2v^{-3}}{\sqrt{v^{-4}}}\right) + \frac{\partial}{\partial u}\left(\frac{0}{\sqrt{v^{-4}}}\right)\right] \\ &= \frac{-1}{2v^{-2}}\left[\frac{\partial}{\partial v}\left(-2v^{-1}\right) + \frac{\partial}{\partial u}(0)\right] \\ &= \frac{-1}{2v^{-2}}\left(2v^{-2}\right) \\ &= -1 \end{aligned}$$

Thus, the curvature of the hyperbolic plane is $K = -1$. That is, the hyperbolic plane is a surface of *constant negative curvature*, and as a result, it cannot be studied as a surface in ordinary 3 dimensional space. Instead, all information about the hyperbolic plane must come from the intrinsic properties derived from its fundamental form.

Exercises

Find the first fundamental form of the given surfaces. Explain the relationship of the fundamental form to the given surface.

1. $\mathbf{r} = \langle u, v, u\rangle$
2. $\mathbf{r} = \langle u, v, u + v\rangle$
3. $\mathbf{r} = \langle v\sin(u), v\cos(u), v\rangle$
4. $\mathbf{r} = \langle v\sin(u), v, v\cos(u)\rangle$
5. $\mathbf{r} = \langle \sin(u)\cos(v), \cos(u), \sin(u)\sin(v)\rangle$
6. $\mathbf{r} = \langle \sin(v)\sin(u), \cos(v)\sin(u), \cos(u)\rangle$
7. $\mathbf{r} = \langle \sin(u)\cosh(v), \sinh(v), \cos(u)\cosh(v)\rangle$
8. $\mathbf{r} = \langle \sin(u)\cosh(v), \sin(u)\sinh(v), \cos(u)\rangle$

Show that each parameterization is orthogonal and then determine the normal curvature of the surface. Then determine the principal curvatures, the Mean curvature, and the Gaussian curvature of the surface. Which surfaces are minimal surfaces? Which are Gaussian flat (i.e, $K = 0$)?

9. $\mathbf{r}(u, v) = \langle v, \sin(u), \cos(u)\rangle$
10. $\mathbf{r}(u, v) = \langle \cos(u), v, \sin(u)\rangle$
11. $\mathbf{r}(u, v) = \langle u, v, u\rangle$
12. $\mathbf{r}(u, v) = \left\langle u, v^2, u\right\rangle$
13. $\mathbf{r}(u, v) = \langle u\cos(v), u\sin(v), u\rangle$
14. $\mathbf{r}(u, v) = \left\langle u^2\cos(v), u^2\sin(v), u\right\rangle$
15. $\mathbf{r}(u, v) = \langle e^u, e^{-u}, v\rangle$
16. $\mathbf{r}(u, v) = \left\langle e^u, e^{-u}, v^2\right\rangle$
17. $\mathbf{r}(u, v) = \langle e^v\sin(u), e^v\cos(u), e^{-v}\rangle$
18. $\mathbf{r}(u, v) = \langle e^u\cos(v), e^u\sin(v), u\rangle$

Use the theorem Egregium to determine the Gaussian curvature of a surface with the given fundamental form.

19. $ds^2 = du^2 + e^{4u}dv^2$
20. $ds^2 = v^2du^2 + v^2dv^2$
21. $ds^2 = v^2du^2 + dv^2$
22. $ds^2 = 4u^2du^2 + 8v^2dv^2$
23. $ds^2 = v^4du^2 + \left(4v^2 + 1\right)dv^2$
24. $ds^2 = du^2 + \sec^2(v)\,dv^2$

25. Find the normal curvature and principle curvatures of $z = x^2 - y^2$ at $(0, 0, 0)$. Does the function have an extremum or a saddle point at $(0, 0, 0)$?

26. Suppose a surface has a fundamental form of

$$ds^2 = du^2 + e^{2ku} dv^2$$

where k is a constant. What is the Gaussian curvature of the surface?

27. In latitude-longitude coordinates, a sphere of radius R centered at the origin has a fundamental form of

$$ds^2 = R^2 d\varphi^2 + R^2 \cos^2(\varphi)\, d\theta^2$$

Apply the Theorem Egregium to this form to determine the curvature of the sphere intrinsically.

28. Show that the surface of revolution

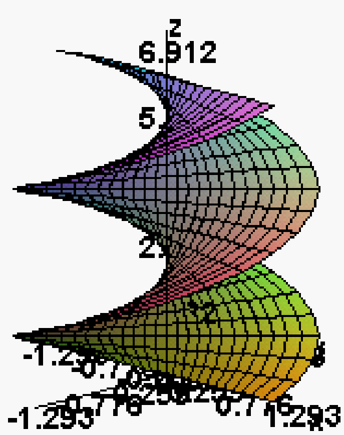
$$\mathbf{r}(u, v) = \left\langle u, \sqrt{R^2 - u^2}\cos(v), \sqrt{R^2 - x^2}\sin(v) \right\rangle$$

is a sphere of radius R. Determine the fundamental form of the surface and use it to find K.

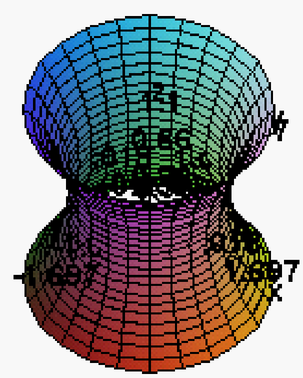
29. The *helicoid* is the surface parametrized by

$$\mathbf{r}(u, v) = \langle \sinh(v)\cos(u), \sinh(v)\sin(u), u\rangle$$

What is its fundamental form?



Ex. 41: A Helicoid



Ex. 42: A Catenoid

30. A *catenoid* is the surface parametrized by

$$\mathbf{r}(u,v) = \langle \cosh(v)\cos(u), \cosh(v)\sin(u), v\rangle$$

Show that it has the same fundamental form as the helicoid (exercise 29). What does this mean?

31. Show that the helicoid is a minimal surface.

32. Show that the catenoid is a minimal surface.

33. The surface of revolution of $y = f(x)$ about the x-axis can be parameterized by

$$\mathbf{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u)\rangle$$

Find the fundamental form and then use the Theorem Egregium to show that the curvature of a surface of revolution is given by

$$K = \frac{-f''(v)}{f(v)\left(1+[f'(v)]^2\right)}$$

34. The *pseudosphere* is a surface of revolution parameterized by

$$\mathbf{r}(u,v) = \left\langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u) + \ln\left[\tan\left(\frac{u}{2}\right)\right]\right\rangle$$

Determine the fundamental form and then use the Theorem Egregium to show that $K = -1$.

35. Euler's Formula: Show that if κ_1 occurs along $\mathbf{r}_u$ (as it does along the cylinder, for instance), then

$$\kappa_n(\theta) = \kappa_1\cos^2(\theta) + \kappa_2\sin^2(\theta)$$

36. Write to Learn: Write an essay in which you prove mathematically that a minimal surface that is Gaussian flat must be a region in a plane. Then use sketches and concepts to explain in your own words why Gaussian flat minimal surfaces must be planar.

37. Write to Learn: Write a short essay in which you use the following steps to prove that the Gaussian curvature K satisfies

$$(\mathbf{n}_u \times \mathbf{n}_v)\cdot\mathbf{n} = K\,\|\mathbf{r}_u \times \mathbf{r}_v\|$$

1. (a) Explain why $\mathbf{r}_u \cdot \mathbf{n} = 0$, and then show that this implies that

   $$\mathbf{r}_{uu}\cdot\mathbf{n} = -\mathbf{r}_u\cdot\mathbf{n}_u, \quad \textit{and} \quad \mathbf{r}_{uv}\cdot\mathbf{n} = -\mathbf{r}_u\cdot\mathbf{n}_v$$

   Also, show that $\mathbf{r}_v \cdot \mathbf{n} = 0$ implies that $\mathbf{r}_{vu}\cdot\mathbf{n} = -\mathbf{r}_v\cdot\mathbf{n}_u$ and $\mathbf{r}_{vv}\cdot\mathbf{n} = -\mathbf{r}_v\cdot\mathbf{n}_v$.

   (b) A cross product identity says that if **A**, **B**, **C**, and **D** are vectors, then

   $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})$$

   Apply this identity to the quantity $(\mathbf{n}_u\times\mathbf{n}_v)\cdot(\mathbf{r}_u\times\mathbf{r}_v)$

(c) Use (a) and (b) to show that

$$(\mathbf{n}_u \times \mathbf{n}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (\mathbf{r}_{uu} \cdot \mathbf{n})(\mathbf{r}_{vv} \cdot \mathbf{n}) - (\mathbf{r}_{uv} \cdot \mathbf{n})^2$$

and that $(\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \|\mathbf{r}_u\|^2 \|\mathbf{r}_v\|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2$.

(d) Conclude by explaining why

$$\frac{(\mathbf{n}_u \times \mathbf{n}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v)}{\|\mathbf{r}_u \times \mathbf{r}_v\|^2} = K$$

and then derive the desired result.

38. Show that if we define K to satisfy the relationship

$$\mathbf{n}_u \times \mathbf{n}_v = K \ (\mathbf{r}_u \times \mathbf{r}_v)$$

then K is the Gaussian Curvature of the surface. You may assume that $\mathbf{r}(u,v)$ is orthogonal, and you may want to use the steps (b) and (c) in exercise 35.

39. (Extends Exercise 41 in section 3-2). Stereographic projecton of a sphere of radius R leads to a parameterization of the form

$$\mathbf{r}(u,v) = \left\langle \frac{2Ru}{u^2+v^2+1}, \frac{2Rv}{u^2+v^2+1}, \frac{R(u^2+v^2-1)}{u^2+v^2+1} \right\rangle \tag{7}$$

Show that $\|\mathbf{r}(u,v)\| = R$ for all (u,v). Then show that the fundamental form is conformal, and calculate the curvature of the surface.

40. Find the principle curvatures of the *Enneper Minimal Surface*, which is parameterized by

$$\mathbf{r}(u,v) = \left\langle u - \frac{1}{3}u^3 + uv^2, \frac{1}{3}v^3 - v - u^2 v, u^2 - v^2 \right\rangle$$

Show that it is in fact a minimal surface. Also, show that the parameterization is conformal and use the result to calculate the Gaussian curvature.

41. Any sufficiently differentiable parameterization of a surface can be transformed into a conformal parameterization. Let's explore this idea for *surfaces of revolution* with parameterizations of the form

$$\mathbf{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u) \rangle$$

In particular, show that if $y(v)$ satisfies the differential equation

$$\frac{dy}{dv} = \frac{f(y)}{\sqrt{1 + [f'(y)]^2}}$$

then the parameterization

$$\mathbf{r}(u, y(v)) = \langle y(v), f(y(v))\cos(u), f(y(v))\sin(u) \rangle$$

is a parameterization of the original surface that is also conformal.

42. Use the result in exercise 41 to find a conformal parameterization of the right circular cone, which is a surface of revolution of the form

$$\mathbf{r}(u,v) = \langle v, v\cos(u), v\sin(u)\rangle$$

Then use the result in exercise 39 to compute the Gaussian curvature of the right circular cone.