Coordinate Transformation

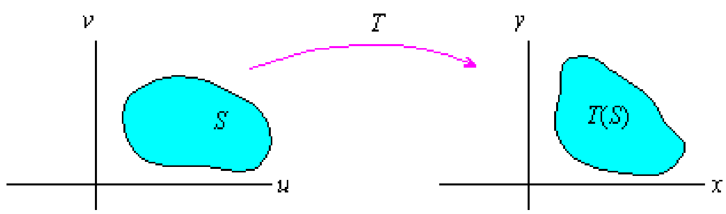
Coordinate Transformations

In this chapter, we explore mappings – where a mapping is a function that "maps" one set to another, usually in a way that preserves at least some of the underlyign geometry of the sets.

For example, a 2-dimensional *coordinate transformation* is a mapping of the form

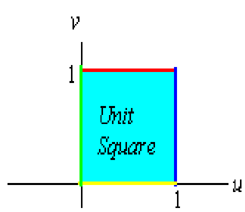
$$T(u,v) = \langle x(u,v), y(u,v) \rangle$$

The functions $x(u,v)$ and $y(u,v)$ are called the *components* of the transformation. Moreover, the transformation T maps a set S in the uv-plane to a set $T(S)$ in the xy-plane:



If S is a region, then we use the components $x = f(u,v)$ and $y = g(u,v)$ to find the image of S under $T(u,v)$.

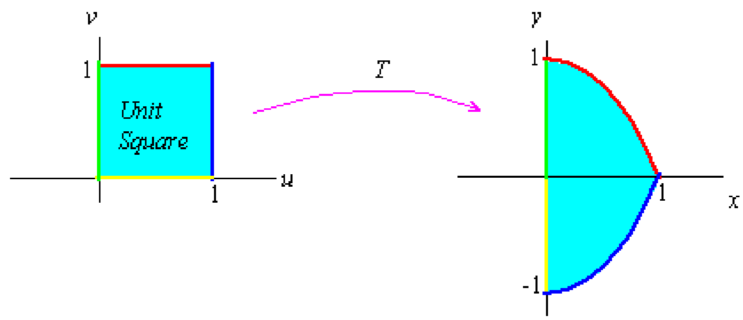
EXAMPLE 1 Find $T(S)$ when $T(u,v) = \left\langle uv, u^2 - v^2 \right\rangle$ and S is the *unit square* in the uv-plane (i.e., $S = [0,1] \times [0,1]$).



Solution: To do so, let's determine the boundary of $T(S)$ in the xy-plane. We use $x = uv$ and $y = u^2 - v^2$ to find the image of the lines bounding the unit square:

Side of Square	Result of $T(u,v)$	Image in xy-plane
$v = 0$, u in $[0,1]$	$x = 0$, $y = u^2$, u in $[0,1]$	y-axis for $0 \leq y \leq 1$
$u = 1$, v in $[0,1]$	$x = v$, $y = 1 - v^2$, v in $[0,1]$	$y = 1 - x^2$, x in $[0,1]$
$v = 1$, u in $[0,1]$	$x = u$, $y = u^2 - 1$, u in $[0,1]$	$y = x^2 - 1$, x in $[0,1]$
$u = 0$, u in $[0,1]$	$x = 0$, $y = -v^2$, v in $[0,1]$	y-axis for $-1 \leq y \leq 0$

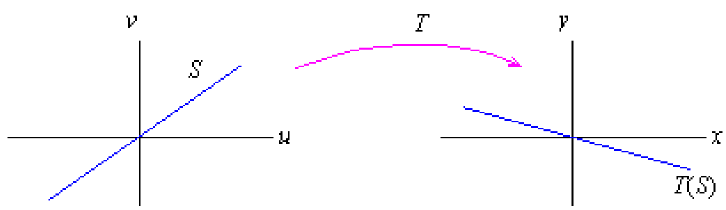
As a result, $T(S)$ is the region in the xy-plane bounded by $x = 0$, $y = x^2 - 1$, and $y = 1 - x^2$.



Linear transformations are coordinate transformations of the form

$$T(u, v) = \langle au + bv, cu + dv \rangle$$

where a, b, c, and d are constants. Linear transformations are so named because they map lines through the origin in the uv-plane to lines through the origin in the xy-plane.



If each point (u, v) in the uv-plane is associated with a column matrix, $[u, v]^t$, then the linear transformation $T(u, v) = \langle au + bv, cu + dv \rangle$ can be written in matrix form as

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

The matrix of coefficients a, b, c, d is called the *matrix of the transformation.*

EXAMPLE 2 Find the image of the unit square under the linear transformation

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Solution: Since linear transformations map straight lines to straight lines, we need only find the images of the 4 vertices of the unit square. To begin with, the point $(0, 0)$ is mapped to $(0, 0)$. Associating the point $(1, 0)$ to the column vector $[1, 0]^t$ yields

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

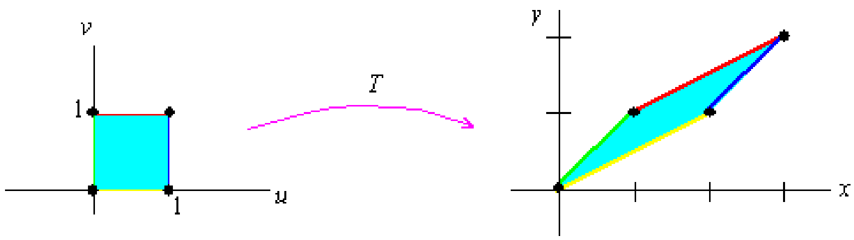
Thus, the point $(1,0)$ is mapped to the point $(3,-2)$. Likewise, associating $(0,1)$ with $[0,1]^t$ leads to

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and associating $(1,1)$ with $[1,1]^t$ leads to

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

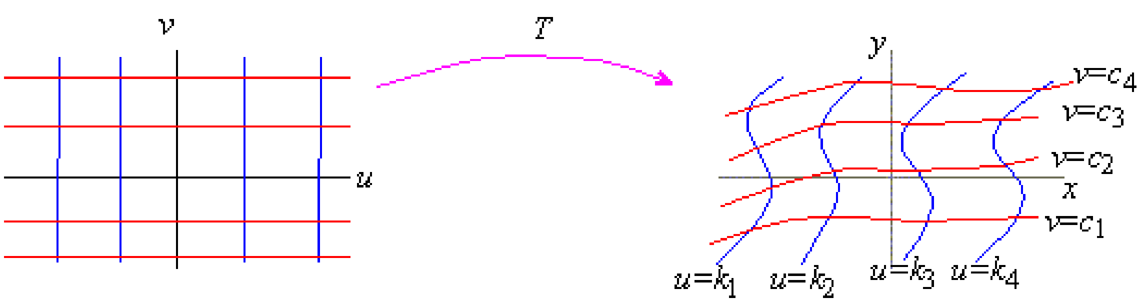
That is, $(0,1)$ is mapped to $(1,1)$ and $(1,1)$ is mapped to $(3,2)$. Thus, the unit square in the uv-plane is mapped to the parallelogram in the xy-plane with vertices $(0,0)$, $(2,1)$, $(1,1)$, and $(3,2)$.



Check your Reading: Is the entire u-axis mapped to 0 by $T(u,v) = \langle v\cos(u), v\sin(u)\rangle$?

Coordinate Systems

Coordinate transformations are often used to define often used to define new *coordinate systems* on the plane. The u-curves of the transformation are the images of vertical lines of the form $u =$ constant and the v-curves are images of horizontal lines of the form $v =$ constant.



Together, these curves are called the *coordinate curves* of the transformation.

EXAMPLE 3 Find the coordinate curves of

$$T(u,v) = \langle uv, u - v^2 \rangle$$

Solution: The u-curves are of the form $u = k$ where k is constant. Thus,

$$x = kv, \quad y = k - v^2$$

so that $v = x/k$ and thus,

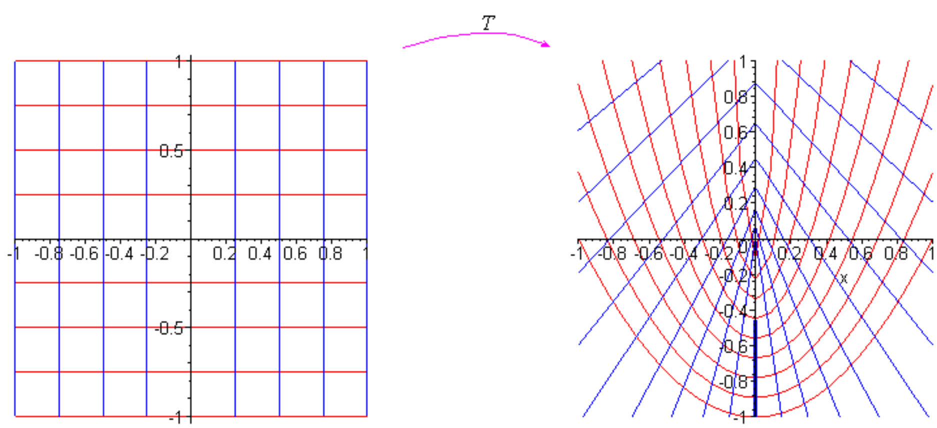
$$y = \frac{-x}{k} + k^2$$

which is a family of straight lines with slope $-1/k$ and intercept k^2.

The v-curves are of the form $v = c$, where c is a constant. Thus, $x = uc$ and $y = u^2 - c$. Since $u = x/c$, the v-curves are of the form

$$y = \frac{x^2}{c^2} - c$$

which is a family of parabolas opening upwards with vertices on the y-axis.



A coordinate transformation $T(u,v)$ is said to be 1-1 on a region S in the uv-plane if each point in $T(S)$ corresponds to only one point in S. The pair (u,v) in S is then defined to be the *coordinates* of the point $T(u,v)$ in $T(S)$.

For example, in the next section we will explore the *polar coordinate transformation*

$$T(r,\theta) = \langle r\cos(\theta), r\sin(\theta) \rangle$$

or equivalently, $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

EXAMPLE 4 What are the coordinate curves of the *polar coordinate transformation*

$$T(r,\theta) = \langle r\cos(\theta), r\sin(\theta)\rangle$$

Solution: The r-curves are of the form $r = R$ for R constant. If $r = R$, then

$$x = R\cos(\theta), \quad y = R\sin(\theta)$$

As a result, $x^2 + y^2 = R^2\cos^2(\theta) + R^2\sin^2(\theta) = R^2$, which is the same as $x^2 + y^2 = R^2$. Thus, the r-curves are circles of radius R centered at the origin.

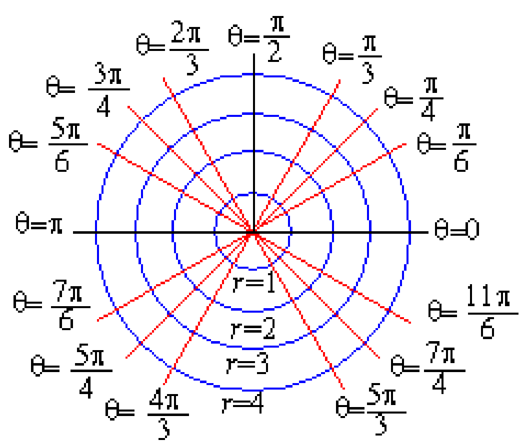
The θ-curves, in which $\theta = c$ for c constant, are given by

$$x = r\cos(c), \qquad y = r\sin(c)$$

As a result, we have

$$\frac{y}{x} = \frac{r\sin(c)}{r\cos(c)} = \tan(c)$$

which is the same as $y = kx$ with $k = \tan(c)$. Thus, the θ-curves are lines through the origin of the xy-plane.



Since θ corresponds to angles, the polar coordinate transformation is not 1-1 in general. However, if we restrict θ to $[0, 2\pi)$ and require that $r > 0$, then the polar coordinate transformation is 1-1 onto the xy-plane *omitting the origin.*

Check your Reading: What point corresponds to $r = 0$ in example 4?

Rotations About the Origin

Rotations about the origin through an angle θ are linear transformations of the form

$$T(u,v) = \langle u\cos(\theta) - v\sin(\theta), u\sin(\theta) + v\cos(\theta)\rangle \tag{1}$$

The matrix of the rotation through an angle θ is given by

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

given that positive angles are those measured counterclockwise (see the exercises).

EXAMPLE 5 Rotate the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$ through an angle $\theta = \pi/3$ about the origin.

Solution: To begin with, the matrix of the rotation is

$$R(\theta) = \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

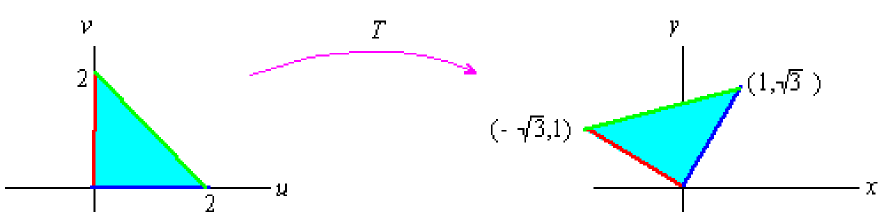
so that the resulting linear transformation is given by

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

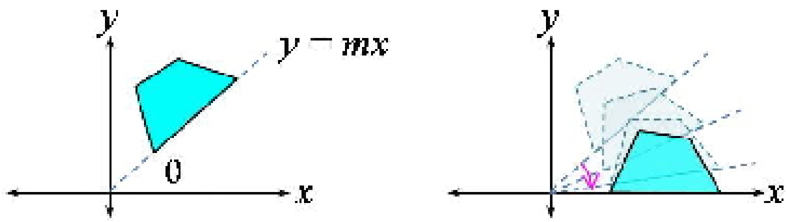
The point $(0,0)$ is mapped to $(0,0)$. The point $(2,0)$ is associated with $[2,0]^t$, so that

$$T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

That is, $(2,0)$ is mapped to $\left(1,\sqrt{3}\right)$. Similarly, it can be shown that $(0,2)$ is mapped to $\left(-\sqrt{3},1\right)$:

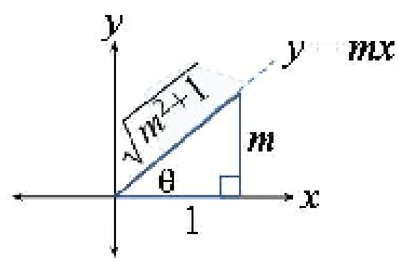


Often rotations are used to put figures into *standard form*, and often this requires rotating a line $y = mx$ onto the x-axis.



If we notice that $m = \tan(\theta)$, then it follows that

$$\cos(\theta) = \frac{1}{\sqrt{m^2+1}}$$
$$\sin(\theta) = \frac{m}{\sqrt{m^2+1}}$$



Thus, the rotation matrix for rotating the x-axis to the line $y = mx$ is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \frac{1}{\sqrt{m^2+1}} \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \qquad (2)$$

Conversely, rotation through an angle $-\theta$ will rotate $y = mx$ to the x-axis (and corresponds to using $-m$ in place of m in (2)).

blue*EXAMPLE 6* blackRotate the triangle with vertices at $(0,0)$, $(1,2)$, and $(-4,2)$ so that one edge lies along the x-axis.

Solution: The line through $(0,0)$ and $(1,2)$ is $y = 2x$, which implies that the rotation matrix is

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \frac{1}{\sqrt{2^2+1}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

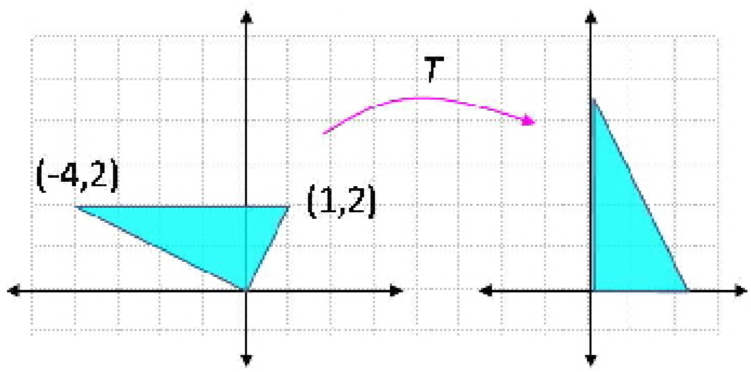
Thus, the point $(1,2)$ is mapped to

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

while the point $(-1,4)$ is mapped to

$$T\left(\begin{array}{c}-4\\2\end{array}\right)=\frac{1}{\sqrt{5}}\left[\begin{array}{cc}1&2\\-2&1\end{array}\right]\left[\begin{array}{c}-4\\2\end{array}\right]=\left[\begin{array}{c}0\\2\sqrt{5}\end{array}\right]$$

Notice that this reveals that that the triangle is a right triangle.



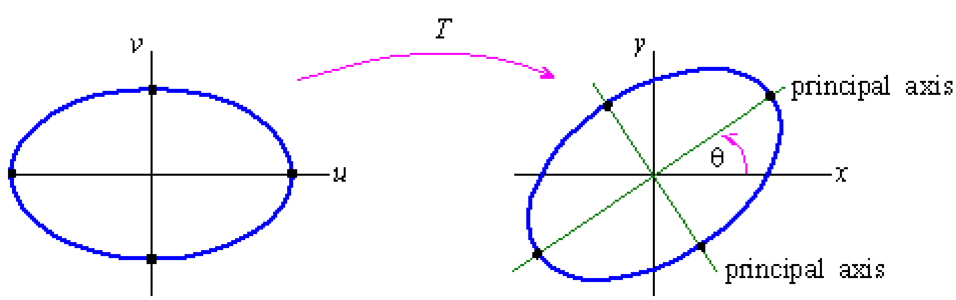
Check your Reading: Why do all linear transformations map $(0,0)$ to $(0,0)$?

Rotation of Conics into Standard Form

If A, B, and C are constants, then the level curves of

$$Q(x,y)=Ax^2+Bxy+Cy^2 \tag{3}$$

are either lines, circles, ellipses, or hyperbolas. If $B\neq 0$, then a curve (**??**) is the image *under rotation* of a conic in standard position in the uv-plane.



Specifically, (**??**) is the image of a conic in standard position in the uv-plane of a rotation

$$\left[\begin{array}{c}x\\y\end{array}\right]=\left[\begin{array}{cc}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{array}\right]\left[\begin{array}{c}u\\v\end{array}\right] \tag{4}$$

that maps the u-axis to a *principal axis* of the conic, which is a line $y = mx$ containing the points closest to or furthest from the origin.

Thus, Lagrange multipliers can be used to determine the equation $y = mx$ of a principal axis, after which replacing x and y by the rotation transformation implied by (2) and (4) will rotate a conic (**??**) into standard form.

blue*EXAMPLE 7* blackRotate the following conic into standard form:

$$5x^2 - 6xy + 5y^2 = 8 \tag{5}$$

Solution: Our goal is to find the extrema of the square of the distance from a point (x, y) to the origin, which is $f(x, y) = x^2 + y^2$, subject to the constraint (5) The associated Lagrangian is

$$L(x, y, \lambda) = x^2 + y^2 - \lambda\left(5x^2 - 3xy + 5y^2 - 21\right)$$

Since $L_x = 2x - \lambda(10x - 3y)$ and $L_y = 2y - \lambda(-3x + 10y)$, we must solve the equations

$$2x = \lambda(10x - 6y), \quad 2y = \lambda(-6x + 10y)$$

Since λ cannot be zero since $(0, 0)$ cannot be a critical point, we eliminate λ using the ratio of the two equations:

$$\frac{2x}{2y} = \frac{\lambda(10x - 6y)}{\lambda(-6x + 10y)} \quad or \quad \frac{x}{y} = \frac{10x - 6y}{-6x + 10y}$$

Cross-multiplication yields $10xy - 6x^2 = 10xy - 6y^2$ so that $y^2 = x^2$. Thus, the principal axes – i.e., the lines containing the extrema – are $y = x$ and $y = -x$.
Using $y = x$ means $m = 1$ and correspondingly,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

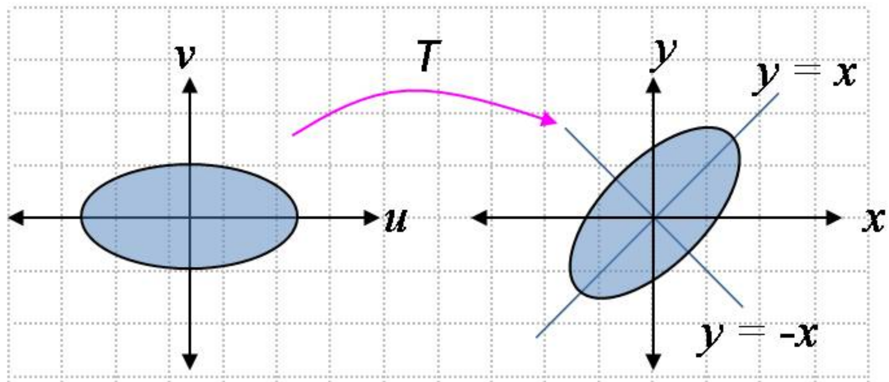
That is, $x = (u - v)/\sqrt{2}$ and $y = (u + v)/\sqrt{2}$, which upon substitution into (5) yields

$$\begin{aligned} 5\left(\frac{u-v}{\sqrt{2}}\right)^2 - 6\left(\frac{u-v}{\sqrt{2}}\right)\left(\frac{u+v}{\sqrt{2}}\right) + 5\left(\frac{u+v}{\sqrt{2}}\right)^2 &= 8 \\ \frac{5\left(u^2 - 2uv + v^2\right)}{2} - \frac{6\left(u^2 - v^2\right)}{2} + \frac{5\left(u^2 + 2uv + v^2\right)}{2} &= 8 \\ 5u^2 + 5v^2 - 6u^2 + 6v^2 + 5u^2 + 5v^2 &= 16 \\ 4u^2 + 16v^2 &= 16 \end{aligned}$$

Consequently, the ellipse (5) is a rotation of the ellipse

$$\frac{u^2}{4} + \frac{v^2}{1} = 16$$

as is shown below:



Exercises:

Find the image in the xy-plane of the given curve in the uv-plane under the given transformation. If the transformation is linear, identify it as such and write it in matrix form.

1. $T(u,v) = \langle u, v^2 \rangle$, $v = 2u$
2. $T(u,v) = \langle uv, u+v \rangle$, $v = 3$
3. $T(u,v) = \langle u-2v, 2u+v \rangle$, $v = 0$
4. $T(u,v) = \langle u+3, v+2 \rangle$, $u^2+v^2 = 1$
5. $T(u,v) = \langle 4u, 3v \rangle$, $u^2+v^2 = 1$
6. $T(u,v) = \langle u^2+v, u^2-v \rangle$, $v = u$
7. $T(u,v) = \langle u^2-v^2, 2uv \rangle$, $v = 1-u$
8. $T(u,v) = \langle u^2-v^2, 2uv \rangle$, $v = u$
9. $T(r,\theta) = \langle r\cos(\theta), r\sin(\theta) \rangle$, $r = 1$
10. $T(r,\theta) = \langle r\cos(\theta), r\sin(\theta) \rangle$, $\theta = \pi/4$

Find several coordinate curves of the given transformation (e.g., $u = -1, 0, 1, 2$ and $v = -1, 0, 1, 2$). What is the image of the unit square under the given transformation? If the transformation is linear, identify it as such and write it in matrix form.

11. $T(u,v) = \langle u+1, v+5 \rangle$
12. $T(u,v) = \langle 2u+1, 3v-2 \rangle$
13. $T(u,v) = \langle -v, u \rangle$
14. $T(u,v) = \langle u+v+1, v+2 \rangle$
15. $T(u,v) = \langle u^2-v^2, 2uv \rangle$
16. $T(u,v) = \langle u^2+v, u^2-v \rangle$
17. $T(u,v) = \langle 2u+3v, -3u+2v \rangle$
18. $T(u,v) = \left\langle \frac{u+v}{2}, \frac{u-v}{2} \right\rangle$
19. Rotation about the origin through an angle $\theta = \frac{\pi}{4}$
20. Rotation about the origin through an angle $\theta = \frac{2\pi}{3}$
21. $T(u,v) = \langle\rangle$
22. $T(u,v) = \langle u\cos(\pi v), u\sin(\pi v) \rangle$
23. $T(r,\theta) = \langle e^r\cos(\theta), e^r\sin(\theta) \rangle$
24. $T(r,t) = \langle r\cosh(t), r\sinh(t) \rangle$

Find a conic in standard form that is the pullback under rotation of the given curve.

25.	$5x^2 + 6xy + 5y^2 = 8$	26.	$5x^2 - 6xy + 5y^2 = 9$
27.	$xy = 1$	28.	$xy = 4$
29.	$7x^2 + 6\sqrt{3}\, xy + 13y^2 = 16$	30.	$13x^2 - 6\sqrt{3}\, xy + 7y^2 = 16$
31.	$52x^2 - 72xy + 73y^2 = 100$	31.	$73x^2 + 72xy + 52y^2 = 100$

33. The conic section

$$x^2 + 2xy + y^2 - x + y = 2$$

is not centered at the origin. Can you rotate it into standard position?

34. The conic section

$$5x^2 + 6xy + 5y^2 - 4x + 4y = -2$$

is not centered at the origin. Can you rotate it into standard position?

35. Show that if

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

then we must also have

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What is the significance of this result?

36. Use matrix multiplication to show that a rotation through angle θ followed by a rotation through angle ϕ is the same as a single rotation through angle $\theta + \phi$.

37. The *parabolic coordinate system* on the xy-plane is the image of the coordinate transformation

$$T(u, v) = \left\langle u^2 - v^2, 2uv \right\rangle$$

Determine the coordinate curves of the transformation, and sketch a few for specific values of u and v.

38. The *tangent coordinate system* on the xy-plane is the image of the coordinate transformation

$$T(u, v) = \left\langle \frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right\rangle$$

Determine the coordinate curves of the transformation, and sketch a few for specific values of u and v.

39. The *elliptic coordinate system* on the xy-plane is the image of the coordinate transformation

$$T(u, v) = \langle \cosh(u) \cos(v), \sinh(u) \sin(v) \rangle$$

Determine the coordinate curves of the transformation, and sketch a few for specific values of u and v.

40. The *bipolar coordinate system* on the xy-plane is the image of the coordinate transformation

$$T(u,v) = \left\langle \frac{\sinh(v)}{\cosh(v)-\cos(u)}, \frac{\sin(u)}{\cosh(v)-\cos(u)} \right\rangle$$

Determine the coordinate curves of the transformation, and sketch a few for specific values of u and v.

41. Write to Learn: A coordinate transformation $T(u,v) = \langle f(u,v), g(u,) \rangle$ is said to be *area preserving* if the area of the image of any region S in the uv-plane is the same as the area of R. Write a short essay explaining why a rotation through an angle θ is area preserving.

42. Write to Learn (Maple): A coordinate transformation $T(u,v) = \langle f(u,v), g(u,) \rangle$ is said to be *conformal* (or *angle-preserving*) if the angle between 2 lines in the uv-plane is mapped to the same angle between the image lines in the xy-plane. Write a short essay explaining why a linear transformation with a matrix of

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

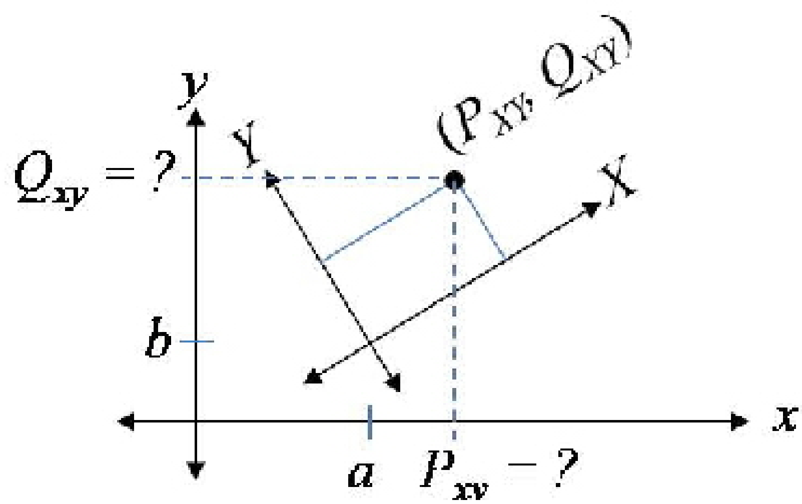
is a conformal transformation.

43. Write to Learn: What type of coordinate system is implied by the coordinate transformation

$$T(u,v) = \langle u, F(u) + v \rangle?$$

What are the coordinate curves? What is significant about tangent lines to these curves? Write a short essay which addresses these questions.

44. Write to Learn: Suppose that we are working in an XY-coordinate system that is centered at (p,q) and is at an angle θ to the x-axis in an xy-coordinate system.



Write a short essay explaining how one would convert coordinates with respect to the XY axes to coordinates in the xy-coordinate system.