Practice Test

Chapter 2 Name ____________________

Instructions. Show your work and/or explain your answers.

1. Find the domain of the function

$$f(x,y) = \sqrt{y} + \sqrt{x^2 - 1}$$

Is the domain open, closed, or neither? Bounded or unbounded? Connnected or not connected?

Solution: $dom(f) = \{(x,y) \mid y \geq 0 \text{ and } x^2 \geq 1\} = \{(x,y) \mid x \leq -1,\ y \geq 0\} \cup \{(x,y) \mid x \geq 1,\ y \geq$ Since domain contains its boundaries $y = 0$, $x = -1$, and $x = 1$, it is *closed.* Since x and y can approach infinity within the domain, it is *unbounded.* Since no path exists from $\{(x,y) \mid x \leq -1,\ y \geq 0\}$ to $\{(x,y) \mid x \geq 1,\ y \geq 0\}$ that stays in the domain, the domain is *not connected.*

2. Show the following limit does not exist by showing that different paths through the origin lead to different limits:

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 - y^2}$$

Solution: Along $y = 0$, the limit is 1. Along $x = 0$, the limit is -1.

3. Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

Solution: No. Along $x = 0$ and $y = 0$, the limit is 1. However, along $y = x$ the limit is 2.

4. Find the linearization of $f(x,y) = x + e^{xy}$ at $(1,0)$

Solution: $f_x(x,y) = 1 + ye^{xy}$, $f_y(x,y) = xe^{xy}$. Thus, $L(x,y) = 2 + 1(x-1) + 1(y-1)$.

5. Find the second order derivatives of

$$f(x,y) = x^2 + e^{xy}$$

Solution: $f_x = 2x + ye^{xy}$, $f_y = xe^{xy}$, $f_{xx} = 2 + y^2e^{xy}$, $f_{yy} = x^2e^{xy}$, $f_{xy} = e^{xy} + xye^{xy}$

6. Find the separated solution of
$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=u$$
Solution: $u(x,t)=\phi(x)T(t)$ implies that $\phi T'+\phi' T=\phi T$, so that
$$\frac{T'}{T}+\frac{\phi'}{\phi}=1, \quad \frac{T'(t)}{T(t)}=\frac{\phi'(x)}{\phi(x)}+1$$
Thus, $T'(t)=-kT(t)$ and $\phi'(x)=(-1-k)\phi(x)$, so that the separated solution is
$$\phi(x)T(t)=Pe^{-kt}e^{(-k-1)x}$$

7. Find $\partial_u z$ when $z=x^2+y^3$ and $x=u^2+uv$, $y=u^3v$

Solution: The chain rule implies that
$$\begin{aligned}\frac{\partial z}{\partial u} &= 2x\frac{\partial x}{\partial u}+3y^2\frac{\partial y}{\partial u}\\ &= 2\left(u^2+uv\right)(2u+v)+3\left(u^3v\right)^2\left(3u^2v\right)\\ &= 4u^3+6u^2v+2uv^2+9u^8v^3\end{aligned}$$

8. Prove that the derivative of a sum is the sum of the derivatives by applying the chain rule for 2 variables to
$$w=x+y$$
where $x=f(t)$ and $y=g(t)$.

Solution: The chain rule implies that
$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$$
However, $w_x=1$ and $w_y=1$, and also $w=f(t)+g(t)$, so that
$$\frac{d}{dt}(f(t)+g(t))=\frac{dw}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=f'(t)+g'(t)$$
thus completing the proof.

9. Find the gradient of the function $g(x,y)=x^2+y^2$, and then show that it is normal to the curve
$$x^2+y^2=25$$
at the point $(3,4)$.

Solution: The curve $x^2+y^2=25$ is a circle. $\nabla g=\langle 2x,2y\rangle$, so $\nabla g(3,4)=\langle 6,8\rangle$. However, $\nabla g(3,4)=\langle 6,8\rangle$ is parallel to the radius $\langle 3,4\rangle$ and thus must be perpendicular to the tangent line.

10. In what direction is the function $f(x, y) = x^2 + y^3$ **decreasing** the fastest at the point $(1, 3)$?

    Solution: The gradient of f is $\nabla f = \langle 2x, 3y^2 \rangle$, so that $\nabla f(1, 3) = \langle 2, 27 \rangle$. This is the direction in which f is *increasing* the fastest. The direction f is *decreasing* the fastest is thus

    $$-\nabla f(1, 3) = \langle -2, -27 \rangle$$

11. Find the extrema and saddle points of $f(x, y) = x^2 + 3xy + 2y^2 - 4x - 5y$.

    Solution: $f_x = 2x + 3y - 4$, $f_y = 3x + 4y - 5$. Thus,

    $$\begin{aligned} 2x + 3y &= 4 \\ 3x + 4y &= 5 \end{aligned}$$

    Thus, the critical point is $(-1, 2)$. Moreover, $f_{xx} = 2$, $f_{xy} = 3$, and $f_{yy} = 4$, so that

    $$D = f_{xx} f_{yy} - (f_{xy})^2 = 4 \cdot 2 - 3^2 = -1$$

    and thus there is a critical point at the saddle.

12. Find the extrema and saddle points of $f(x, y) = 4x^3 - 6x^2 y + 3y^2$

    Solution: $f_x = 12x^2 - 12xy$, $f_y = -6x^2 + 6y$, Thus, $12x^2 = 12xy$ and $6x^2 = 6y$. Since $x^2 = xy$, $x = 0$ or $x = y$. If $x = 0$, then $x^2 = y$ implies that $y = 0$, and the critical point is $(0, 0)$. If $x = y$, then $x^2 = y$ implies that $x^2 = x$ or $x = 1, 0$. Thus, the critical points are $(0, 0)$ and $(1, 1)$. However,

    $$D = (24x - 12y)\, 6 - (12x)^2 = 144x - 72y - 144x^2$$

    Thus, $D(0, 0) = 0$ and there is no info, and $D(1, 1) = 144 - 72 - 144 = -72 < 0$, so there is a saddle at $(1, 1)$.

13. Find the point(s) on the curve $xy = 1$ that are closest to the origin.

    Solution: That is, minimize $f(x, y) = x^2 + y^2$ subject to $xy = 1$. If $g(x, y) = xy$, then $\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \langle y, x \rangle$, so that

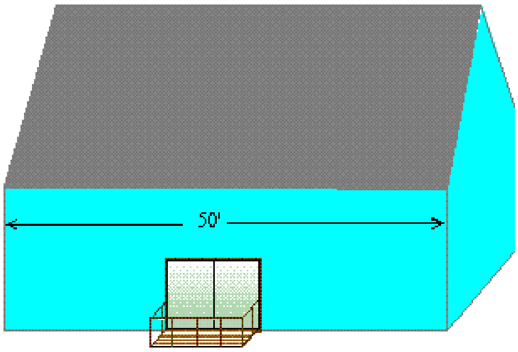
    $$2x = \lambda y, \quad 2y = \lambda x$$

    Since neither x, y can be zero, we have $\lambda = 2x/y$, so that

    $$2y = \frac{2x}{y} x \quad y^2 = x^2 \quad y = x, y = -x$$

    If $y = -x$, then $xy = -x^2 = 1$ which has no solution. If $y = x$, then $xy = x^2 = 1$, so $x = 1, -1$ and the critical points are $(1, 1)$ and $(-1, -1)$. In both cases $f(1, 1) = f(-1, -1) = 2$. Moreover, $f(2, 1/2) = 4.25$, so we must have minima at these points.

14. Use Lagrange Multipliers to solve the following: John wants to build a 500 ft^2 deck behind his house.



His house is 50 feet long, and correspondingly, he wants the deck to be between 5 and 50 feet long. What dimensions of the deck will minimize the lengths of the rail around the 3 exposed sides of the deck?

Solution: Let x be the length and y be the width of the deck. Then $xy = 500$. Let L denote the length of the rail. Then

$$L = x + 2y$$

Thus, we must minimize $L = x+2y$ subject to $xy = 500$ for x in $[5, 50]$. If $g(x, y) = xy$, then $\nabla L = \langle 1, 2 \rangle$ and $\nabla g = \langle y, x \rangle$, so that

$$1 = \lambda y, \qquad 2 = \lambda x$$

Since $\lambda = 1/y$, substitution leads to

$$2 = \frac{1}{y} x \qquad \textit{and} \qquad 2y = x$$

Substituting $x = 2y$ into the constraint yields $2y^2 = 500$, or

$$y^2 = 250, \qquad y = \sqrt{250} = 5\sqrt{10}$$

If $x = 2y$ and $y = 5\sqrt{10}$, then $x = 10\sqrt{10}$, so that $\left(10\sqrt{10}, 5\sqrt{10}\right)$ is a critical point. At that point

$$L = 10\sqrt{10} + 2 \cdot 5\sqrt{10} = 20\sqrt{10} = 63.25'$$

When $x = 5$, then $y = 100$ and at $(5, 100)$ the length is

$$L = 5 + 2 \cdot 100 = 205$$

When $x = 50$, then $y = 10$ and at $(50, 10)$, the length is

$$L = 50 + 2 \cdot 10 = 70'$$

Thus, the shortest rail occurs when $x = 10\sqrt{10} = 31.622'$ and $y = 5\sqrt{10} = 15.811'$

15. ** Heating of a 2 dimensional surface (such as in a sheet of metal) is modeled by the *2 dimensional heat equation*

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} + k^2 \frac{\partial^2 u}{\partial y^2}$$

where $u(x, y, t)$ is a function of 3 variables and k is a constant. What is the separable solution of the 2 dimensional heat equation (hint: involves 2 separation constants)?

Solution: Let $u(x, y, t) = \phi(x) \rho(y) T(t)$. Then

$$\phi(x) \rho(y) T'(t) = k^2 \phi''(x) \rho(y) T(t) + k^2 \phi(x) \rho''(y) T(t)$$

Dividing through by $\phi(x) \rho(y) T(t)$ leads to

$$\frac{\phi(x) \rho(y) T'(t)}{\phi(x) \rho(y) T(t)} = \frac{k^2 \phi''(x) \rho(y) T(t)}{\phi(x) \rho(y) T(t)} + \frac{k^2 \phi(x) \rho''(y) T(t)}{\phi(x) \rho(y) T(t)}$$

which simplifies to

$$\frac{T'(t)}{T(t)} = \frac{k^2 \phi''(x)}{\phi(x)} + \frac{k^2 \rho''(y)}{\rho(y)}$$

Both sides of the equation must be constant, so that

$$\frac{T'(t)}{T(t)} = -\omega^2 \qquad \textit{and} \qquad \frac{k^2 \phi''(x)}{\phi(x)} + \frac{k^2 \rho''(y)}{\rho(y)} = -\omega^2$$

The second equation can now be written as

$$\frac{k^2 \phi''(x)}{\phi(x)} = -\omega^2 - \frac{k^2 \rho''(y)}{\rho(y)}$$

thus implying both sides of this equation are constant (we let $-\lambda^2$ denote this constant).

$$\frac{k^2 \phi''(x)}{\phi(x)} = -\lambda^2 \qquad \textit{and} \qquad -\omega^2 - \frac{k^2 \rho''(y)}{\rho(y)} = -\lambda^2$$

The last equation becomes

$$\omega^2 \rho(y) + k^2 \rho''(y) = \lambda^2 \rho(y) \qquad \textit{or} \quad \rho''(y) + \frac{\omega^2 - \lambda^2}{k^2} \rho(y) =$$

If $\omega^2 > \lambda^2$, then the equation is a harmonic oscillator and has a solution of

$$\rho(y) = A_1 \cos\left(\frac{y\sqrt{\omega^2 - \lambda^2}}{k}\right) + B_1 \sin\left(\frac{y\sqrt{\omega^2 - \lambda^2}}{k}\right)$$

If $\omega^2 = \lambda^2$, then $\rho''(y) = 0$ and $\rho(y) = A_1 + B_1 y$. If $\omega^2 < \lambda^2$, then

$$\rho(y) = A_1 \cosh\left(\frac{y\sqrt{\omega^2 - \lambda^2}}{k}\right) + B_1 \sinh\left(\frac{y\sqrt{\omega^2 - \lambda^2}}{k}\right)$$

Moreover,

$$\frac{k^2\phi''(x)}{\phi(x)} = -\lambda^2 \ \ \textit{implies that } \phi''(x) + \frac{\lambda^2}{k^2}\phi(x) = 0$$

implies that

$$\phi(x) = \rho(y) = A_2 \cos\left(\frac{\lambda}{k}x\right) + B_2 \sin\left(\frac{\lambda}{k}x\right)$$

and finally, $T'(t) = -\omega^2 T(t)$ implies that $T(t) = Pe^{-\omega^2 t}$. Thus, for $\omega^2 > \lambda^2$, the separated solution is

$$u(x,y,t) = Pe^{-\omega^2 t}\left(A_2 \cos\left(\frac{\lambda}{k}x\right) + B_2 \sin\left(\frac{\lambda}{k}x\right)\right)\left(A_1 \cos\left(\frac{y\sqrt{\omega^2-\lambda^2}}{k}\right) + B_1 \sin\left(\frac{y\sqrt{\omega^2-\lambda^2}}{k}\right)\right)$$

and similar for the other two cases.