Triple Integrals in Cylindrical Coordinates

Many applications involve densities for solids that are best expressed in non-Cartesian coordinate systems. In particular, there are many applications in which the use of triple integrals is more natural in either cylindrical or spherical coordinates.

For example, suppose that $f(r,\theta) \geq g(r,\theta)$ in polar coordinates and that $U(x,y,z)$ is a continuous function. If S is the solid between $z = f(x,y)$ and $z = g(x,y)$ over a region R in the xy-plane, then

$$\int\int\int_S U(x,y,z)\,dV = \iint_R \left[\int_{g(x,y)}^{f(x,y)} U(x,y,z)\,dz\right] dA$$

Let's suppose now that in *polar coordinates,* R is bounded by $\theta = \alpha$, $\theta = \beta$, $r = p(\theta)$, and $r = q(\theta)$. Since $dA = r dr d\theta$ in polar coordinates, a change of variables into *cylindrical coordinates* is given by

$$\int\int\int_S U(x,y,z)\,dV = \int_\alpha^\beta \int_{p(\theta)}^{q(\theta)} \int_{f(r,\theta)}^{g(r,\theta)} U(r\cos\theta, r\sin(\theta), z)\;\; r\; dz dr d\theta \quad (1)$$

In practice, however, it is often more straightforward to simply evaluate the first integral in z and then transform the resulting double integral into polar coordinates.

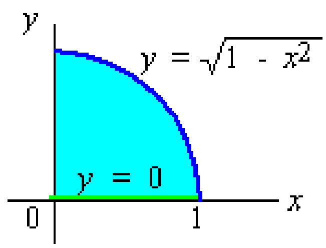
EXAMPLE 1 Evaluate the following

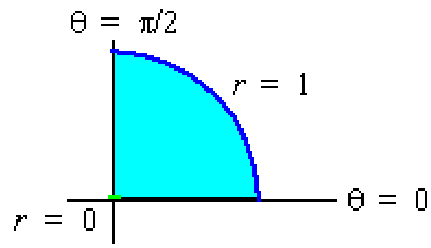
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2xy} \left(x^2+y^2\right) dz dy dx$$

Solution: Rather than employ (1) directly, let's first evaluate the integral in z. That is,

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2xy} \left(x^2+y^2\right) dz dy dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(zx^2+zy^2\right)\Big|_0^{2xy} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} 2xy\left(x^2+y^2\right) dy dx \\ &= \int\int_R 2xy\left(x^2+y^2\right) dA\end{aligned}$$

where R is the quarter of the unit circle in the $1st$ quadrant.





In polar coordinates, R is bounded by $\theta = 0$, $\theta = \pi/2$, $r = 0$, and $r = 1$. Thus,

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2xy} \left(x^2 + y^2\right) dz dy dx &= \int_0^{\pi/2} \int_0^1 2r\cos(\theta)\, r\sin(\theta)\, r^2\; r dr d\theta \\
&= \int_0^{\pi/2} \int_0^1 \sin(2\theta)\, r^5\; dr d\theta \\
&= \int_0^{\pi/2} \sin(2\theta) \left.\frac{r^6}{6}\right|_0^1 d\theta \\
&= \int_0^{\pi/2} \frac{1}{6} \sin(2\theta)\, d\theta \\
&= \frac{1}{6}
\end{aligned}$$

Check your Reading: Where did the r come from in (1)?

Triple Integrals in Spherical Coordinates

If $U(r, \theta, z)$ is given in cylindrical coordinates, then the spherical transformation

$$z = \rho\cos(\phi), \quad r = \rho\sin(\phi)$$

transforms $U(r, \theta, z)$ into $U(\rho\sin(\phi), \theta, \rho\cos(\phi))$. Similar to polar coordinates, we have

$$\frac{\partial(z, r)}{\partial(\rho, \phi)} = \rho d\rho d\phi$$

so that a triple integral in cylindrical coordinates becomes

$$\int_\alpha^\beta \int_{p(\theta)}^{q(\theta)} \int_{f(r,\theta)}^{g(r,\theta)} U(r, \theta, z)\; r\; dz dr d\theta = \int_\alpha^\beta \int_{p(\theta)}^{q(\theta)} \int_{f(\rho\sin(\phi),\theta)}^{g(\rho\sin(\phi),\theta)} U(\rho\sin(\phi), \theta, \rho\cos(\phi))\; r\rho\; d\rho d\phi\, d\theta$$

However, $r = \rho\sin(\phi)$, which leads to the following:

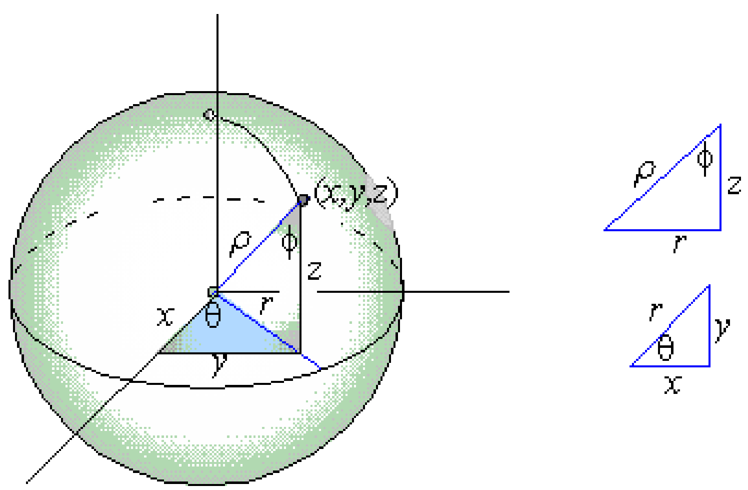
Triple Integrals in Spherical Coordinates: If S is a solid bounded in spherical coordinates by $\theta = \alpha$, $\theta = \beta$, $\phi = p(\theta)$, $\phi = q(\theta)$, $\rho = f(\phi, \theta)$, and $\rho = g(\phi, \theta)$, and if $U(\rho, \phi, \theta)$ is continuous on S, then

$$\int\int\int_S U(\rho,\phi,\theta)\,dV = \int_\alpha^\beta \int_{p(\theta)}^{q(\theta)} \int_{f(\phi,\theta)}^{g(\phi,\theta)} U(\rho,\phi,\theta)\ \rho^2 \sin(\phi)\ d\rho d\phi d\theta \tag{2}$$

In particular, it is important to notice that

$$dV = \rho^2 \sin(\phi)\ d\rho d\phi d\theta$$

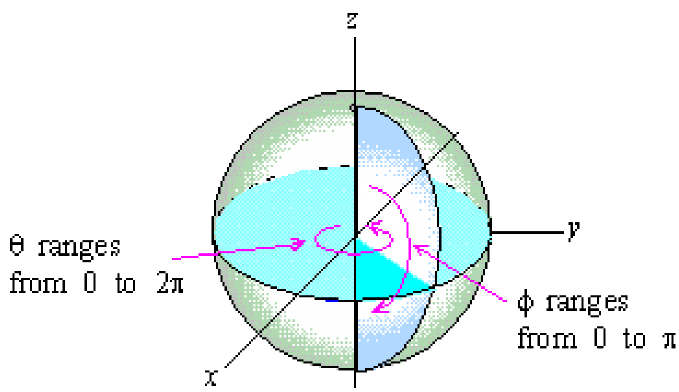
and it is acceptable to use $dV = r\rho d\rho d\phi d\theta$ since $r = \rho \sin(\phi)$. It is also important to remember the relationships given by the two right triangles relating (x, y, z) to (ρ, ϕ, θ).



In particular, $x^2 + y^2 = r^2$ implies $x^2 + y^2 = \rho^2 \sin^2(\phi)$ and $r^2 + z^2 = \rho^2$ implies that

$$x^2 + y^2 + z^2 = \rho^2 \tag{3}$$

Also remember that θ ranges over $[0, 2\pi]$, while ϕ ranges over $[0, \pi]$.



EXAMPLE 2 Use spherical coordinates to evaluate the triple integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dzdydx$$

Solution: To begin with, we notice that this iterated integral reduces to

$$\int\int\int_{\substack{unit \\ sphere}} \sqrt{x^2+y^2+z^2} dV$$

As a result, in spherical coordinates it becomes

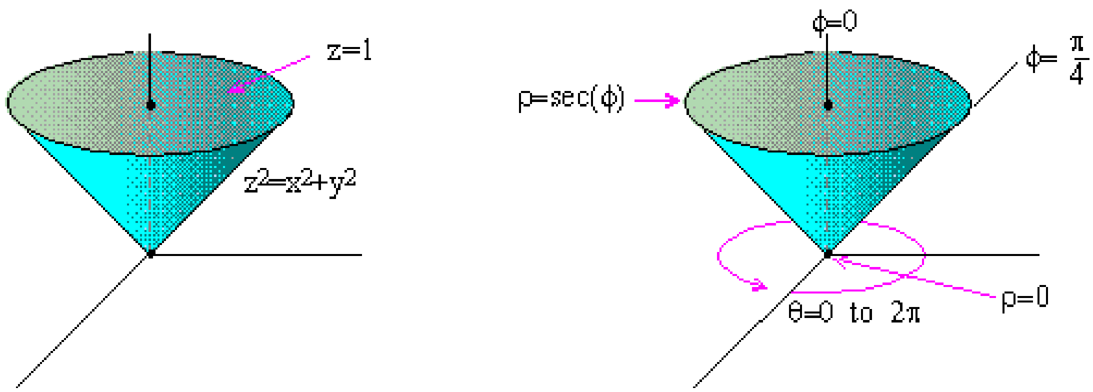
$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \sqrt{\rho^2}\, \rho^2 \sin(\phi)\, d\rho d\phi d\theta$$

since $x^2 + y^2 + z^2 = \rho^2$. Thus, we have

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3 \sin(\phi)\, d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^3 \sin(\phi)\, d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \left.\frac{\rho^4}{4}\right|_0^1 \sin(\phi)\ d\phi d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi} \sin(\phi)\ d\phi d\theta \\
&= \frac{1}{4}\int_0^{2\pi} -\cos(\phi)|_0^{\pi}\, d\theta \\
&= \pi
\end{aligned}$$

EXAMPLE 3 Find the volume of the solid above the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$.

Solution: The cone $z^2 = x^2 + y^2$ corresponds to $\phi = \pi/4$ in spherical.



Moreover, $z = 1$ corresponds to $\rho\cos(\phi) = 1$, or $\rho = \sec(\phi)$. Thus,

$$\begin{aligned} V = \int\int\int_S dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec(\phi)} \rho^2 \sin(\phi)\, d\rho d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4} \left.\frac{\rho^3}{3}\right|_0^{\sec(\phi)} \sin(\phi)\, d\phi d\theta \\ &= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \sec^3(\phi)\sin(\phi)\, d\phi d\theta \end{aligned}$$

However, $\sec(\phi)\sin(\phi) = \tan(\phi)$, so that

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \tan(\phi)\sec^2(\phi)\, d\phi d\theta$$

Thus, $u = \tan(\phi)$, $du = \sec^2(\phi)\, d\phi$, $u(0) = \tan(0) = 0$ and $u(\pi/4) = \tan(\pi/4) = 1$ yields

$$\begin{aligned} V &= \frac{1}{3}\int_0^{2\pi}\int_0^1 u du d\theta = \\ &= \frac{1}{3}\int_0^{2\pi}\frac{1}{2}d\theta \\ &= \frac{\pi}{3} \end{aligned}$$

Check your Reading: Why does the cone $z^2 = x^2+y^2$ correspond to $\phi = \pi/4$?

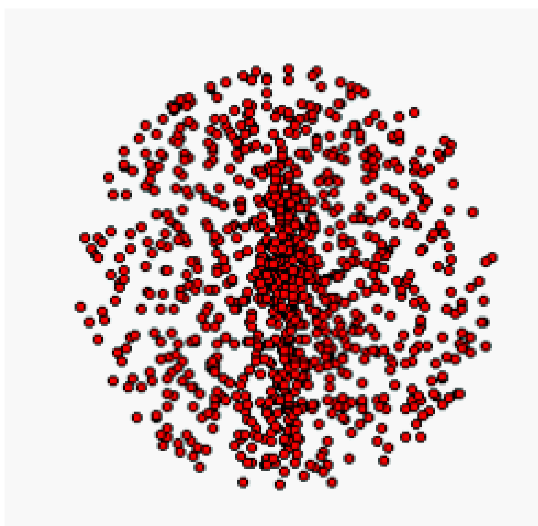
Applications in Spherical and Cylindrical Coordinates

Triple integrals in spherical and cylindrical coordinates occur frequently in applications. For example, it is not common for charge densities and other real-world distributions to have *spherical symmetry,* which means that the density is a function only of the distance ρ. (Note: Scientists and engineers use ρ both to denote charge density and also to denote distance in spherical coordinates. The context in which ρ appears will indicate how it is being used).

EXAMPLE 4 The charge density for a certain charge cloud contained in a sphere of radius 10 cm centered at the origin is given by

$$\rho(x, y, z) = 100\sqrt{x^2 + y^2 + z^2} \ \frac{\mu C}{cm^3}$$

What is the total charge contained within a sphere? (μC = micro-coulombs)



Solution: If Ω denotes the solid sphere of radius 10 cm centered at the origin, then the total charge is

$$Q = \int\int\int_{\Omega} 100\sqrt{x^2 + y^2 + z^2} \, dV$$

However, $x^2 + y^2 + z^2 = \rho^2$ leads to

$$Q = 100 \int_0^{2\pi} \int_0^{\pi} \int_0^{10} \rho \ \ \rho^2 \sin(\phi) \ \ d\rho d\phi d\theta$$

: $2500 \sin \phi$ (i.e., charge density is proportional to ρ). Evaluation of the integral leads to

$$\begin{aligned} Q &= 100 \int_0^{2\pi} \int_0^{\pi} \left. \frac{\rho^4}{4} \right|_0^{10} \sin(\phi) \; d\rho d\phi d\theta \\ &= 100 \int_0^{2\pi} \int_0^{\pi} 2500 \sin(\phi) \, d\phi d\theta \\ &= 100 \int_0^{2\pi} 5000 d\theta \\ &= 1,000,000\pi \; \mu C \end{aligned}$$

which is $Q = \pi$ coulombs.

Triple integrals in spherical and cylindrical coordinates are common in the study of electricity and magnetism. In fact, quantities in the fields of electricity and magnetism are often defined in spherical coordinates to begin with.

EXAMPLE 5 The power emitted by a certain antenna has a power density per unit volume of

$$p(\rho, \phi, \theta) = \frac{P_0}{\rho^2} \sin^4(\phi) \cos^2(\theta)$$

where P_0 is a constant with units in Watts. What is the total power within a sphere of radius 10 m?

Solution: The total power P will satisfy

$$\begin{aligned} P &= \int\int\int_{\Omega} \frac{P_0}{\rho^2} \sin^4(\phi) \cos^2(\theta) \, dV \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{10} \frac{P_0}{\rho^2} \sin^4(\phi) \cos^2(\theta) \quad \rho^2 \sin(\phi) \, d\rho d\phi d\theta \\ &= P_0 \int_0^{2\pi} \int_0^{\pi} \int_0^{10} \sin^4(\phi) \; \sin(\phi) \; \cos^2(\theta) \quad d\rho d\phi d\theta \\ &= 10 P_0 \int_0^{2\pi} \int_0^{\pi} \left(1 - \cos^2(\phi)\right)^2 \; \sin(\phi) \; \cos^2(\theta) \quad d\phi d\theta \end{aligned}$$

Let us now let $u = \cos(\phi)$, $du = -\cos(\phi) \, d\phi$, $u(0) = 1$, and $u(\pi) = -1$. Then

$$\begin{aligned} P &= -10 P_0 \int_0^{2\pi} \int_1^{-1} \left(1 - u^2\right)^2 \; \cos^2(\theta) \quad du d\theta \\ &= 10 P_0 \int_0^{2\pi} \frac{16}{15} \cos^2 \theta d\theta \end{aligned}$$

However, $2\cos^2(\theta) = \cos(2\theta) + 1$, so that

$$P = \frac{80}{25}P_0 \int_0^{2\pi} (\cos(2\theta) + 1)\, d\theta = \frac{32\pi}{5} P_0 \, Watts$$

Check your Reading: Why does the cone $z^2 = x^2 + y^2$ correspond to $\phi = \pi/4$?

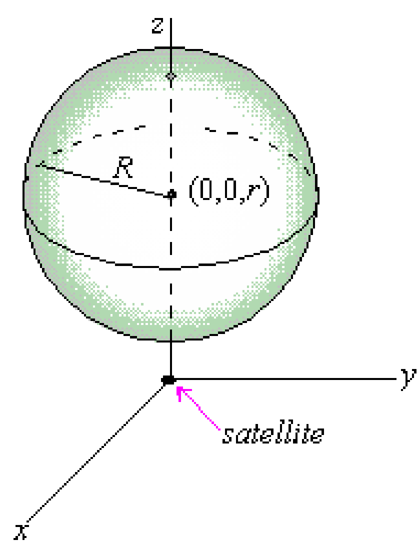
The Inverse Square Law

Suppose two point masses with masses m and M respectively are located a distance r apart. Sir Isaac Newton's *inverse square law* states that the magnitude $|F|$ of the gravitational force between the two point masses is

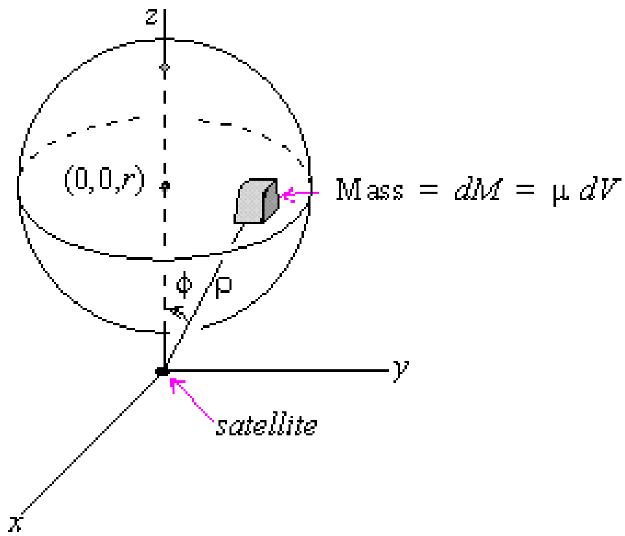
$$|F| = G\frac{Mm}{r^2} \tag{4}$$

where G is the universal gravitational constant. However, as Newton realized and struggled with for some time, objects in the real world are not point-masses and instead, the law (4) might need to be modified.

In particular, let's suppose that one of the bodies is not a "point-mass," but instead is a sphere of radius R with uniform mass density μ. For $r > R$ constant, let's suppose that the sphere is centered at $(0, 0, r)$. If the other body is a point-mass "satellite" of mass m located at the origin, then the gravitational force is directed along the z-axis.



Suppose now that a small "piece" of the sphere is located at a point (ρ, ϕ, θ) (in spherical coordinates), and suppose that it has a small mass dM. Then the distance between the small piece and the origin is ρ.



so that by (4) the "small" magnitude $d|F|$ of the gravitational force between the small "piece" and the satellite is

$$d|F| = \frac{-Gm\,dM}{\rho^2} \tag{5}$$

The amount of $d|F|$ in the vertical direction is then given by $\cos(\phi)\,d|F|$ (see above).

Thus, the total gravitational force in the vertical direction is

$$|F| = \int\int\int_S \cos(\phi)\,d|F| = \int\int\int_S \frac{-Gm\ \cos(\phi)}{\rho^2} dM$$

where S is the sphere corresponding to the "planet". If dV denotes the volume of a small "piece" of the sphere, then $dM = \mu dV$, which leads to

$$|F| = -Gm\mu \int\int\int_S \frac{\cos(\phi)}{\rho^2} dV \tag{6}$$

In Cartesian coordinates, the sphere S is given by

$$x^2 + y^2 + (z-r)^2 = R^2 \qquad or \qquad x^2 + y^2 + z^2 - 2rz + r^2 = R^2$$

In spherical coordinates this becomes

$$\rho^2 - 2r\rho\cos(\phi) + r^2 - R^2 = 0$$

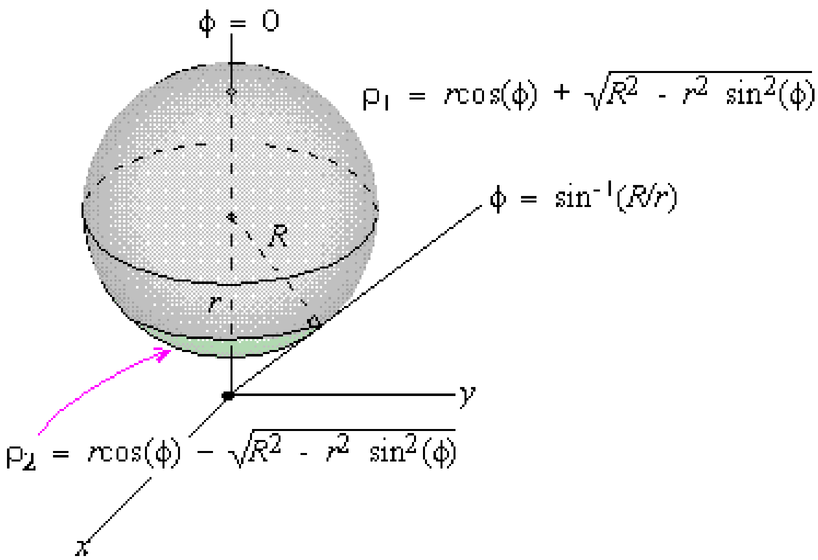
which by the quadratic formula leads to

$$\begin{aligned}\rho &= r\cos(\phi) \pm \sqrt{R^2 - r^2\left(1 - \cos^2(\phi)\right)} \\ &= r\cos(\phi) \pm \sqrt{R^2 - r^2\sin^2(\phi)}\end{aligned}$$

Thus, the sphere is contained between

$$\rho_1 = r\cos\phi - \sqrt{R^2 - r^2\sin^2(\phi)} \ \ and \ \ \rho_2 = r\cos\phi + \sqrt{R^2 - r^2\sin^2(\phi)}$$

Let us also note that ϕ ranges from 0 to $\sin^{-1}(R/r)$ while θ ranges over $[0, 2\pi]$.



Evaluating (6) in spherical coordinates leads to

$$\begin{aligned} |F| &= -Gm\mu \int_0^{2\pi}\int_0^{\sin^{-1}(R/r)}\int_{\rho_1}^{\rho_2} \frac{\cos(\phi)}{\rho^2}\,\rho^2\sin(\phi)\,d\rho d\phi d\theta \\ &= -Gm\mu \int_0^{2\pi}\int_0^{\sin^{-1}(R/r)}\int_{\rho_1}^{\rho_2} \cos(\phi)\sin(\phi)\,d\rho d\phi d\theta \\ &= -Gm\mu \int_0^{2\pi}\int_0^{\sin^{-1}(R/r)} (\rho_2 - \rho_1)\cos(\phi)\sin(\phi)\,d\rho d\phi d\theta \end{aligned}$$

Since $\rho_2 - \rho_1 = 2\left(R^2 - r^2\sin^2(\phi)\right)^{1/2}$, this in turn leads to

$$|F| = -2Gm\mu \int_0^{2\pi}\int_0^{\sin^{-1}(R/r)} \left(R^2 - r^2\sin^2(\phi)\right)^{1/2}\sin(\phi)\cos(\phi)\,d\phi d\theta$$

If we let $u(\phi) = R^2 - r^2\sin^2(\phi)$, then the limits of integration become

$$u(0) = R^2 \qquad and \qquad u\left(\sin^{-1}\left(\frac{R}{r}\right)\right) = R^2 - r^2\left(\frac{R^2}{r^2}\right) = 0$$

Moreover, $du = -2r^2 2\sin(\phi)\cos(\phi)\,d\phi$, so that

$$\begin{aligned}
|F| &= \frac{Gm\mu}{r^2}\int_0^{2\pi}\int_0^{\sin^{-1}(R/r)} \left(R^2 - r^2\sin^2(\phi)\right)^{1/2} \left(-2r^2\sin(\phi)\cos(\phi)\right) d\phi d\theta \\
&= \frac{Gm\mu}{r^2}\int_0^{2\pi}\int_{R^2}^{0} u^{1/2} du\, d\theta \\
&= \frac{Gm\mu}{r^2}\int_0^{2\pi} \left.\frac{u^{3/2}}{3/2}\right|_{R^2}^{0} d\theta \\
&= \frac{-Gm\mu}{r^2}\int_0^{2\pi} \frac{2R^3}{3} d\theta \\
&= \frac{-Gm}{r^2}\frac{4\pi R^3\mu}{3}
\end{aligned}$$

However, the volume of the sphere is $V = 4\pi R^3/3$, so that the mass of the sphere is $M = \mu V = \mu 4\pi R^3/3$. Thus, we have shown that

$$|F| = \frac{-GMm}{r^2}$$

That is, a uniformly-dense spherical "planet" of mass M and a point-mass of mass M at the center of the sphere have the same gravitational attraction on a "satellite" point mass outside the sphere. Since the electromagnetic force also satisfies an inverse square law, this result also says that the electromagnetic force between spheres with uniform charge density is equivalent to the electromagnetic force between point-charges.

Exercises

Convert to cylindrical coordinates and evaluate:

1. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^2 z\, dzdydx$
2. $\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^2 z\, dzdydx$
3. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 \left(\sqrt{x^2+y^2}+z\right) dzdydx$
4. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 2z\sqrt{x^2+y^2}\, dzdydx$
5. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 \frac{dzdydx}{x^2+y^2+1}$
6. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_0^1 \frac{zdzdydx}{x^2+y^2+1}$
7. $\int_0^1\int_0^x\int_0^{|x|+1} \frac{dz\, dydx}{|x|+1}$
8. $\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}} \frac{dz\, dydx}{\sqrt{x^2+y^2}}$
9. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{x^2+y^2} (z+1)^2\, dzdydx$
10. $\int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-y^2}} z\, dzdydx$

Evaluate the following triple integrals using spherical coordinates.

11. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{x^2+y^2+z^2}$

12. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{zdzdydx}{x^2+y^2+z^2}$

13. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{xdzdydx}{x^2+y^2+z^2}$

14. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{ydzdydx}{x^2+y^2+z^2}$

15. $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \frac{dzdydx}{\sqrt{x^2+y^2}}$

16. $\int_{0}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} \frac{dzdydx}{\sqrt{y^2+z^2}}$

17. $\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \left(x^2+y^2\right) dzdydx$

18. $\int_{0}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} x^2 dzdydx$

Identify the solid, and then find its volume.

19. $\rho = 0$ to $\rho = 1$
$\phi = 0$ to $\phi = \pi$
$\theta = 0$ to $\theta = 2\pi$

20. $\rho = 1$ to $\rho = 2$
$\phi = 0$ to $\phi = \pi$
$\theta = 0$ to $\theta = 2\pi$

21. $\rho = 0$ to $\rho = 1$
$\phi = 0$ to $\phi = \frac{\pi}{4}$
$\theta = 0$ to $\theta = 2\pi$

22. $\rho = 0$ to $\rho = 1$
$\phi = 0$ to $\phi = \pi$
$\theta = 0$ to $\theta = \pi$

23. below $x^2 + y^2 + z^2 = 1$
above $x^2 + y^2 = z^2$

24. inside $x^2 + y^2 = 1$
between $z = 0$ and $z = 1$

The following are volume charge densities of charge clouds contained in a sphere of radius 1 meter. Calculate the total charge inside the sphere. Consider ρ_0 to be a constant.

25. $\rho(x,y,z) = 2\ C/m^3$

26. $\rho(x,y,z) = 4\ C/m^3$

27. $\rho(x,y,z) = \rho_0\sqrt{x^2+y^2+z^2}\ C/m^3$

28. $\rho(x,y,z) = \frac{\rho_0}{\sqrt{x^2+y^2+z^2}}\ C/m^3$

29. $\rho(x,y,z) = \rho_0 e^{-\sqrt{x^2+y^2+z^2}}\ C/m^3$

30. $\rho(x,y,z) = \rho_0 \frac{e^{-\left(x^2+y^2+z^2\right)}}{\sqrt{x^2+y^2+z^2}}$

31. The solid cone between the xy-plane and the right circular cone $(z-1)^2 = x^2 + y^2$ has a volume charge density of

$$\rho(x,y,z) = 1 - \left(x^2+y^2\right) z^2$$

What is the total charge contained inside the solid cone?

32. Suppose that two concentric spheres of radius a and b, respectively, with $b > a$ are centered at the origin, and suppose that the volume charge density between the two spheres is

$$\rho(\rho,\phi,\theta) = \frac{\rho_0 (b-a) z^2}{\left(x^2+y^2+z^3\right)^{5/2}}$$

with ρ_0 constant. What is the total charge between the two spheres?

33. A certain sphere of radius 1 meter centered at the origin has a mass density of

$$\mu(x, y, z) = \sqrt{x^2 + y^2 + z^2}\ \frac{kg}{m^3}$$

What is the mass of the sphere?

34. Suppose that the solid S is the "spherical cap" between $x^2 + y^2 + z^2 = 2$ and $z = 1$ if the mass density is

$$\mu(x, y, z) = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

35. What is the center of mass of the hemisphere $x^2 + y^2 + z^2 = R^2$ with $z \geq 0$ if the mass-density μ of the hemisphere is constant?

36. What is the center of mass of the solid above $x^2 + y^2 = z^2$ and below $x^2 + y^2 + z^2 = 1$ if the mass-density μ is constant?

37. In example 6 of section 6, it is shown that the gravitational potential between a mass m located at the point $(0, 0, r)$ and a sphere of radius R centered at the origin with a constant mass density μ is given by

$$U = -Gm \int\int\int_S \frac{\mu dV}{\sqrt{x^2 + y^2 + (z - r)^2}}$$

where S is the sphere. Convert to triple integrals and evaluate for $r > R$ to show that a sphere with uniform mass density has the same potential as a point mass, namely,

$$U = \frac{-GmM}{r}$$

38. What is the gravitational potential of a sphere of radius R with uniform mass-density if $r < R$ (that is, when the satellite is inside the earth)?

39. Write to Learn: The right circular cone with height h and base with radius R is the solid below the plane $z = h$ and above the cone $R^2 z^2 = h^2\left(x^2 + y^2\right)$. In a short essay, show that the cone corresponds to

$$\phi = \tan^{-1}\left(\frac{R}{h}\right)$$

and then use integration in spherical coordinates to find its volume.

40. Write to Learn: In a short essay, explain why if $f(x, y, z)$is a function only of the distance of a point (x, y, z) from the origin—that is, if

$$f(x, y, z) = f\left(\sqrt{x^2 + y^2 + z^2}\right)$$

for all (x, y, z)—and if S is a sphere of radius R centered at the origin, then

$$\int\int\int_S f(x, y, z)\, dV = 4\pi \int_0^R f(\rho)\ \rho^2\ d\rho$$