Double Integrals in Polar Coordinates

Part 1: The Area Differential in Polar Coordinates

We can also apply the change of variable formula to the *polar coordinate transformation*

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

However, due to the importance of polar coordinates, we derive its change of variable formula more rigorously.

To begin with, the Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r$$

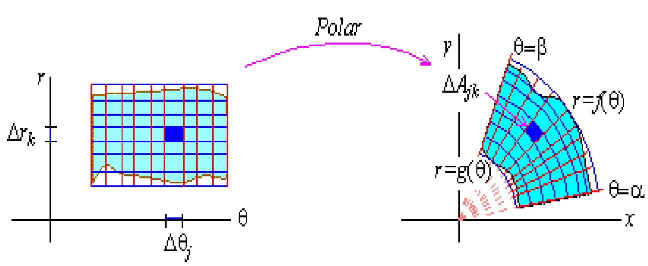
As a result, the area differential for polar coordinates is

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = rdrd\theta$$

Let us consider now the polar region S defined by

$$\theta = \alpha, \theta = \beta, r = g(\theta), r = f(\theta)$$

where $f(\theta)$ and $g(\theta)$ are contained in $[p,q]$ for all θ in $[\alpha,\beta]$. If $\theta_0,\ldots,\theta_m$ is an h-fine partition of $[\alpha,\beta]$ and $r_0,\ldots,r_n$ is an h-fine partition of $[p,q]$, then the image of $[\alpha,\beta]\times[p,q]$ is a partition of the image of the region with near parallelograms whose areas are denoted by ΔA_{jk}:



Since the area differential is $dA = rdrd\theta$, the area of the "near parallelogram" is approximately

$$\Delta A_{jk} \approx r_j \Delta r_j \Delta\theta_j$$

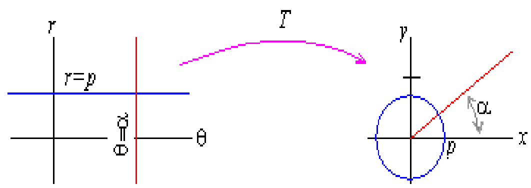
so that if $x_{jk} = r_j\cos(\theta_k)$ and $y_{jk} = r_j\sin(\theta_k)$, then

$$\lim_{h\to 0}\sum_{j=1}^{n}\sum_{k=1}^{m}\phi(x_{jk},y_{jk})\Delta A_{jk} = \lim_{h\to 0}\sum_{j=1}^{n}\sum_{k=1}^{m}\phi(r_j\cos(\theta_k), r_j\sin(\theta_k)) r_j\Delta r_j\Delta\theta_j$$

Writing each of these limits as double integrals results in the formula for *change of variable in polar coordinates*:

$$\iint_R \phi(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} \phi(r\cos(\theta), r\sin(\theta))\ r\,dr\,d\theta \tag{1}$$

To aid in the use of (1), let us notice that if p is constant, then $r = p$ is a circle of radius p centered at the origin in the xy-plane, while if α is constant, then $\theta = \alpha$ is a ray at angle α beginning at the origin of the xy-plane.



Moreover, the origin corresponds to $r = 0$.

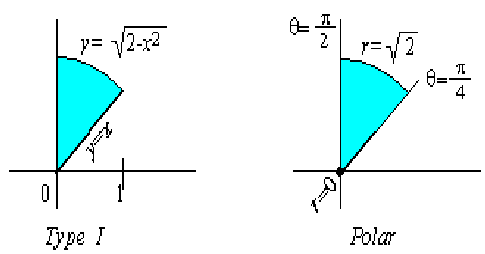
EXAMPLE 1 Use (1) to evaluate

$$\int_0^1 \int_0^{\sqrt{2-x^2}} \left(x^2+y^2\right) dy\,dx$$

Solution: To do so, we transform the iterated integral into a double integral

$$\int_0^1 \int_0^{\sqrt{2-x^2}} \left(x^2+y^2\right) dy\,dx = \iint_R \left(x^2+y^2\right) dA$$

where R is a sector of a circle with radius $\sqrt{2}$. In polar coordinates, R is the region between $r = 0$ and $r = \sqrt{2}$ for θ in $[\pi/4, \pi/2]$:



Since $r^2 = x^2 + y^2$, the double integral thus becomes

$$\iint_R \left(x^2+y^2\right) dA = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r^2\ r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r^3\ dr\,d\theta$$

and the resulting iterated integral is then easily evaluated:

$$\iint_R \left(x^2+y^2\right) dA = \int_{\pi/4}^{\pi/2} \left.\frac{r^4}{4}\right|_0^{\sqrt{2}} d\theta = \int_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{4}$$

Check your Reading: What does $y = x$ correspond to in polar coordinates?

Areas and Volumes in Polar Coordinates

If R is a region in the xy-plane bounded by $\theta = \alpha$, $\theta = \beta$, $r = g(\theta)$, $r = f(\theta)$, then (1) implies that

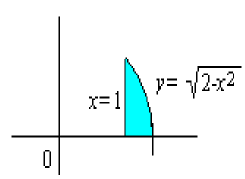
$$Area\ of\ R = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{f(\theta)} r dr d\theta$$

thus allowing us to find areas in polar coordinates.

EXAMPLE 2 Find the area of the region between $x = 1$, $x = \sqrt{2}$, $y = 0$, and

$$y = \sqrt{2 - x^2}$$

Solution: Since $x = 1$ corresponds to $r\cos(\theta) = 1$ or $r = \sec(\theta)$, the region is between the line $r = \sec(\theta)$ and a circle of radius $\sqrt{2}$ from $\theta = 0$ to $\theta = \pi/4$:



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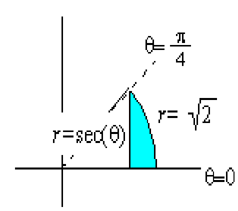


Polar

Thus, the area of the region is

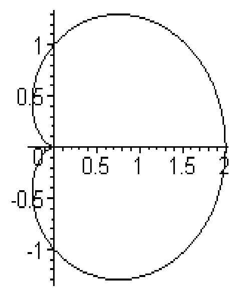
$$Area = \iint_R dA = \int_0^{\pi/4} \int_{\sec(\theta)}^{\sqrt{2}} r dr d\theta$$

and evaluation of the iterated integral leads to

$$\begin{aligned} Area &= \int_0^{\pi/4} \left.\frac{r^2}{2}\right|_{\sec(\theta)}^{\sqrt{2}} d\theta \\ &= \frac{1}{2}\int_0^{\pi/4} \left(2 - \sec^2(\theta)\right) d\theta \\ &= \frac{1}{2}\left(2\theta - \tan(\theta)\right)\Big|_0^{\pi/4} \\ &= \frac{1}{4}\pi - \frac{1}{2} \end{aligned}$$

Moreover, we can use polar coordinates to find areas of regions enclosed by graphs of polar functions.

EXAMPLE 3 What is the area of the region enclosed by the cardioid $r = 1 + \cos(\theta)$, θ in $[0, 2\pi]$.



Solution: Since the cardioid contains the origin, the lower boundary is $r = 0$. Thus, its area is

$$Area = \int_0^{2\pi} \int_0^{1+\cos(\theta)} r dr d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^{1+\cos(\theta)} d\theta$$

Substituting and expanding leads to

$$\begin{aligned} Area &= \frac{1}{2}\int_0^{2\pi} \left[1 + 2\cos(\theta) + \cos^2(\theta)\right] d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left[1 + 2\cos(\theta) + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\right] d\theta \\ &= \frac{1}{2}\left(\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{4}\sin(2\theta)\right)_0^{2\pi} \\ &= \frac{3\pi}{2} \end{aligned}$$

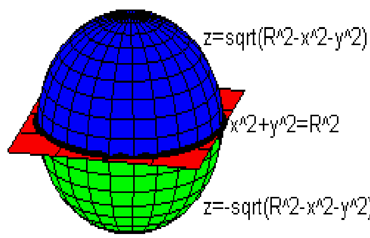
Polar coordinates can also be used to compute volumes. For example, the equation of a sphere of radius R centered at the origin is

$$x^2 + y^2 + z^2 = R^2$$

Solving for z then yields shows us that the sphere can be considered the solid between the graphs of the two functions

$$g(x,y) = -\sqrt{R^2 - x^2 - y^2}, \qquad f(x,y) = \sqrt{R^2 - x^2 - y^2}$$

over the circle $x^2 + y^2 = R^2$ in the xy-plane.



Since circle $x^2 + y^2 = R^2$ defines the type I region

$$\begin{array}{ll} x = -R & y = -\sqrt{R^2 - x^2} \\ x = R & y = \sqrt{R^2 - x^2} \end{array}$$

the volume of the sphere of radius R is given by the iterated integral

$$V = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2 - x^2 - y^2}\, dydx \tag{2}$$

EXAMPLE 4 Use polar coordinates to evaluate

$$V = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2 - x^2 - y^2}\, dydx$$

Solution: To begin with, we rewrite the iterated integral as a double integral over the interior of the circle of radius R centered at the origin, which is often denoted by $\mathbf{D}$:

$$V = \int\int_{\mathbf{D}} 2\sqrt{R^2 - (x^2 + y^2)}\; dA$$

In polar coordinates, the disc $\mathbf{D}$ of radius R is bounded by the curves $\theta = 0$, $\theta = 2\pi$, $r = 0$, $r = R$, so that

$$\begin{aligned} V &= \int\int_{\mathbf{D}} 2\sqrt{R^2 - x^2 - y^2}\, dA \\ &= \int_0^{2\pi} \int_0^R 2\sqrt{R^2 - r^2}\, r dr d\theta \end{aligned}$$

Thus, if we let $u = R^2 - r^2$, then $du = -2rdr$, $u(0) = R^2$, $u(R) = 0$, so that

$$\begin{aligned} V &= -\int_0^{2\pi} \int_{R^2}^{0} u^{1/2}\, du\, d\theta \\ &= -\int_0^{2\pi} \left. \frac{u^{3/2}}{3/2} \right|_{R^2}^{0} d\theta \\ &= \int_0^{2\pi} \frac{2\left(R^2\right)^{3/2}}{3} d\theta \\ &= \frac{4\pi}{3} R^3 \end{aligned}$$

Check your Reading: What is the volume of the unit sphere?

Independent Normal Distributions

In statistics, a *normally distributed random variable* with mean μ and standard deviation σ has a *Gaussian density,* which is function of the form

$$p_1(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\left(2\sigma^2\right)} \tag{3}$$

It follows that the joint density for two independent, normally distributed events is a function of two variables of the form

$$p(x, y) = p_1(x)\, p_2(y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-(x-\mu_1)^2/\left(2\sigma_1^2\right)} e^{-(x-\mu_1)^2/\left(2\sigma_1^2\right)}$$

For simplicity, we will consider here only independent, normally distributed events with mean $\mu = 0$ in both and standard deviations $\sigma = \sigma_1 = \sigma_2$. In such cases, the joint density function is

$$p(x, y) = \frac{1}{2\pi\sigma^2} e^{-\left(x^2+y^2\right)/\left(2\sigma^2\right)}$$

EXAMPLE 5 Let (X, Y) be the coordinates of the final resting place of a ball which is released from a position on the z-axis toward the xy-plane, and suppose the two coordinates are independently normally distributed with a mean of 0 and a standard deviation of

3 feet.

What is the probability that the ball's final resting place will be no more than 5 feet from the origin?

Solution: Since $\sigma = 3$, the joint density function is

$$p(x, y) = \frac{1}{18\pi} e^{-\left(x^2+y^2\right)/18}$$

and we want to know the probability that (X, Y) will be in a circle with radius 5 centered at the origin. Since such a circle corresponds to $r = 0$ to $r = 5$ for θ in $[0, 2\pi]$, the probability is

$$P\left[X^2 + Y^2 \leq 25\right] = \iint_R \frac{1}{18\pi} e^{-\left(x^2+y^2\right)/18} dA$$

Converting to polar coordinates then yields

$$P\left[X^2 + Y^2 \leq 25\right] = \frac{1}{18\pi} \int_0^{2\pi} \int_0^5 e^{-r^2/18} r\,dr\,d\theta$$

and if we now let $u = r^2$, $du = 2rdr$, then $u(0) = 0$ and $u(5) = 25$ implies that

$$\begin{aligned} P\left[X^2 + Y^2 \leq 25\right] &= \frac{1}{36\pi} \int_0^{2\pi} \int_0^{25} e^{-u/18} du\,d\theta \\ &= \frac{1}{36\pi} \int_0^{2\pi} \left(18 - 18e^{-25/18}\right) d\theta \\ &= 1 - e^{-25/18} \\ &= 0.750648 \end{aligned}$$

Thus, there is about a 75% chance that the ball's final resting place will be no more than 5 feet from the origin.

Check your Reading: How exactly do we interpret $P\left[X^2+Y^2 \leq 25\right]$?

An Important Result in Statistics

Finally, the value of the integral

$$I = \int_0^\infty e^{-x^2} dx$$

is very important in statistical applications. To evaluate it, we first notice that

$$I^2 = \left[\int_0^\infty e^{-x^2} dx\right]\left[\int_0^\infty e^{-y^2} dy\right] = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dy dx$$

That is, I^2 is a type I iterated integral which can be converted to polar coordinates.

EXAMPLE 6 Evaluate the integral

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dy dx$$

Solution: To do so, let us notice that

$$I^2 = \int \int_{Quad\ I} e^{-\left(x^2+y^2\right)} dA$$

However, in polar coordinates, the first quadrant is given by $r = 0$ to $r = \infty$ for $\theta = 0$ to $\theta = \pi/2$. Thus,

$$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

As a result, we can write

$$I^2 = \int_0^{\pi/2} \left[\lim_{R\to\infty} \int_0^R e^{-r^2} r dr\right] d\theta$$

Thus, if we let $u = r^2$, $du = 2rdr$, $u(0) = 0$, $u(R) = R^2$, then

$$\begin{aligned} I^2 &= \frac{1}{2}\int_0^{\pi/2}\left[\lim_{R\to\infty}\int_0^{R^2} e^{-u}\,du\right]d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\left[\lim_{R\to\infty}\left(e^0 - e^{-R^2}\right)\right]d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta \\ &= \frac{\pi}{4} \end{aligned}$$

Thus, $I = \sqrt{\pi}/2$, which implies both

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \qquad \textit{and} \qquad \int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$$

Exercises:

Evaluate the following iterated integrals by transforming to polar coordinates.

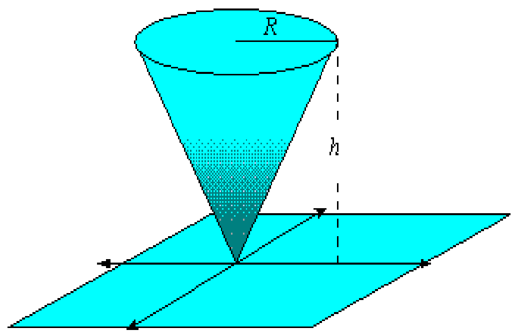
1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\; dydx$
2. $\int_0^1 \int_0^{\sqrt{1-x^2}} \tan^{-1}\left(\frac{y}{x}\right) dydx$
3. $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$
4. $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{x^2+y^2}} dydx$
5. $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{y\,dxdy}{x^2+y^2}$
6. $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{x\,dxdy}{x^2+y^2}$
7. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{9-x^2-y^2}dydx$
8. $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \frac{x^2-y^2}{x^2+y^2} dydx$
9. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\; dydx$
10. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{y}{\sqrt{x^2+y^2}}\; dydx$
11. $\int_0^1 \int_0^x \frac{x}{x^2+y^2}\; dydx$
12. $\int_0^1 \int_0^x \frac{x}{\sqrt{x^2+y^2}}\; dydx$
13. $\int_0^1 \int_1^{\sqrt{2-y^2}} \frac{x}{x^2+y^2} dxdy$
14. $\int_0^1 \int_1^{\sqrt{2-y^2}} \frac{y}{x^2+y^2} dxdy$
15. $\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \frac{dydx}{[x^2+y^2]^{3/2}}$
16. $\int_0^1 \int_{1-y}^{\infty} \frac{dxdy}{[x^2+y^2]^{3/2}}$

Each of the following polar curves encloses a region that contains the origin.

Find the area of the region the curve encloses.

17.	$r = 5,\ \theta\ in\ [0, 2\pi]$	18.	$r = 3,\ \theta\ in\ [-\pi, \pi]$
19.	$r = \sin(\theta), \theta\ in\ [0, \pi]$	20.	$r = 4\cos(\theta), \theta\ in\ [0, \pi]$
21.	$r = \pi\theta - \theta^2, \theta\ in\ [0, \pi]$	22.	$r = \lvert\theta\rvert + 1, \theta\ in\ [-\pi, \pi]$
23.	$r = \sin(3\theta), \theta\ in\ [0, 2\pi/3]$	24.	$r = 4\cos(3\theta), \theta\ in\ [0, 2\pi/3]$
25.	$r = \sin(5\theta), \theta\ in\ [0, 2\pi/5]$	26.	$r = \sin^2(\theta), \theta\ in\ [0, \pi]$
27.	$r = 1 + \cos(2\theta), \theta\ in\ [0, \pi]$	28.	$r = 1 + \sin(3\theta), \theta\ in\ [0, 2\pi/3]$
29.	$r = \sin(\theta) + \cos(\theta), \theta\ in\ [0, \pi]$	30.	$r = 3\sin(\theta) + 4\cos(\theta), \theta\ in\ [0, \pi]$

31. Use polar coordinates to find the volume of a right circular cone with height h and a circular base with radius R



(hint: the equation of the cone is

$$z = \frac{h}{R}\sqrt{x^2 + y^2}$$

32. A right circular cone with a base of radius R is sliced by a plane of the form

$$z = h_1 + (h_2 - h_1)\frac{x + R}{2R}$$

where h_1 and h_2 are positive. What is the shape of the solid between this plane and the xy-plane, and what is its volume?

33. Recall that if $0 < \varepsilon < 1$ and $p > 0$, then

$$r = \frac{p}{1 - \varepsilon\cos(\theta)}$$

is an ellipse which encloses a region R. Evaluate

$$\iint_R \frac{dA}{[x^2 + y^2]^{3/2}}$$

34. Evaluate the double integral

$$\iint_R y\,dA$$

where R is the polar ellipse described in exercise 33.

35. In example 4, what is the probability that

1. (a) The final position of the ball is in the 1st quadrant and is no more than 5 feet from the origin.
   (b) The final position of the ball is between 3 and 7 feet from the origin.
   (c) The final position of the ball is in the xy-plane.

36. After several throws at a dart board, a dart thrower finds that both the X and Y coordinates of his darts have a mean of 0 and a standard deviation of 3 inches. What is the probability that a randomly selected dart throw from all those he has thrown will be in the "bulls eye", if the bulls eye is a circle of radius one inch centered at the origin?

37. Suppose an airplane has two rocket engines whose time of ignition with respect to a "time zero" is normally distributed with a standard deviation of $\sigma = 0.01$ seconds. If the rockets' ignitions are independent events, what is the probability that the sum of the squares of the firing times is less than 0.01?

38. In exercise 37, what is the probability that the left engine will fire no more than $\sqrt{3}$ times later than the right engine?

39. The antennae lengths of a sample of 32 woodlice were measured and found to have a mean of 4 mm and standard deviation of 2.37 mm. Assuming the antennae lengths are normally distributed, what is the probability of one of the antennae of a woodlice being twice as long as the other? (Hint: substitute to translate the means to 0).

40. Acme sheet metal produces several hundred rectangular sheets of metal each day. If errors in the lengths and widths of the rectangular sheets are independent random variables with mean of 0 and a standard deviation of s = 0.1 inches, then what is the probability that the error in the area of the rectangular sheets exceeds 0.1 inches?

41. Use the method in the discussion preceding example 6 to evaluate

$$J = \int_0^\infty x^2 e^{-x^2} dx$$

42. Find the area and the centroid of a cardioid of the form

$$r = 1 + \cos(\theta)$$

43. Write to Learn: A freezer produces ice cubes with normally distributed temperatures with a mean of $0°F$ and a standard deviation of $2°F$. Write a short essay in which you compute and explain the probability that two ice cubes chosen at random will have temperatures that differ by no more than $3°F$, assuming the temperatures are independent.

44. Try it out! Drop a ball several times (i.e., 20-30 times) from a position directly above an "origin" in an xy-plane you create. (Hint: to avoid any bias, you might want to secure the ball with a thread and then release the ball by cutting the thread). Suppose that

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

denotes the final stopping points of the ball. The sample means of both the x's and the y's should be practically zero. The sample standard deviation for the x's is

$$\sigma_x = \sqrt{\frac{\sum_{j=1}^{n} (x_j - \bar{x})}{n}}$$

and the sample standard deviation σ_y for the y's is similar. Show that $\sigma_x \approx \sigma_y$ and then repeat example 5 using the sample standard deviations as the value for σ.