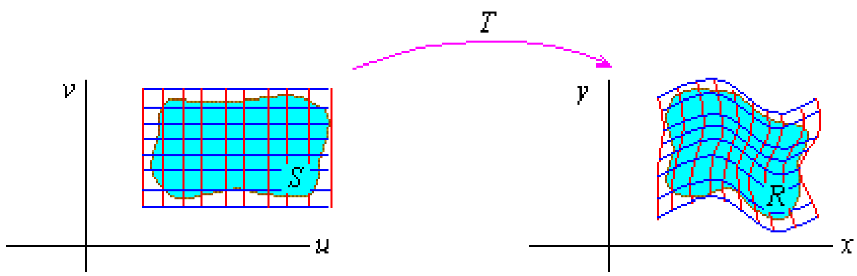
Change of Variables in Double Integrals

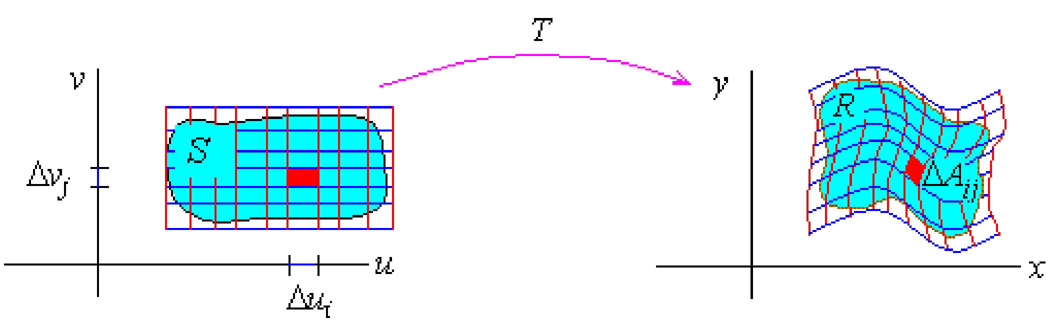
Part 1: Area of the Image of a Region

It is often advantageous to evaluate $\iint_R \phi(x,y)\,dA$ in a coordinate system other than the xy-coordinate system. In this section, we develop a method for converting double integrals into iterated integrals in other coordinate systems.

Let's suppose that $T(u,v) = \langle f(u,v), g(u,v)\rangle$ is a 1-1 map of a region S in the uv-plane onto a region R in the xy-plane, and let's suppose that S is contained in a rectangle $[a,b]\times[c,d]$ on which both f and g are differentiable.



A partition of $[a,b]$ into n subintervals of width Δu_i and $[c,d]$ into m subintervals of width Δv_j covers S with a collection of rectangles. If we require $\Delta u_i < h$ and $\Delta v_j < h$ for some sufficiently small $h > 0$, then the image of the rectangles under T is a collection of regions covering R which are approximately parallelograms.



If ΔA_{ij} is the area of the image of the ij^{th} rectangle, $i = 1,\ldots,n$, and $j = 1,\ldots,m$, then the area of R is approximately

$$Area\ of\ R \approx \sum\sum \Delta A_{ij}$$

However, in chapter 3, we learned that the area of the image of a rectangle is approximately

$$\Delta A_{ij} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_i^*,v_j^*)} \Delta u_i \Delta v_j$$

where $\left(u_i^*, v_j^*\right)$ is a point in the ij-th rectangle. Thus, the area of R is approximately

$$Area\ of\ R \approx \sum\sum \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{\left(u_i^*,v_j^*\right)} \Delta u_i \Delta v_j$$

and in the limit as h approaches 0, the double sum leads to a double integral. Thus, we have

$$Area\ of\ R = \int\int_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dA_{uv}$$

where $dA_{uv} = dvdu$ if S is type I and $dA_{uv} = dudv$ if S is type II.

EXAMPLE 1 Find and describe the image of $[0, 2\pi] \times [0, 1]$ under the transformation

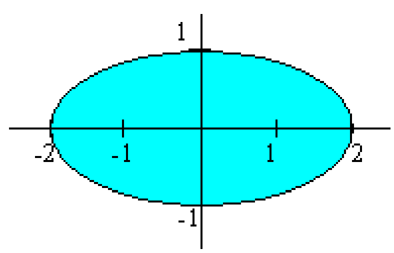
$$T(u,v) = \langle 2v\cos(u), v\sin(u)\rangle$$

and then find the area of that image.

Solution: Since $x = 2v\cos(u)$ and $y = v\sin(u)$, we have

$$\frac{x^2}{4} + \frac{y^2}{1} = v^2\cos^2(u) + v^2\sin^2(u) = v^2$$

which because u ranges from 0 to 2π is an ellipse centered at the origin. Moreover, these ellipses completely cover the region inside the ellipse corresponding to $v = 1$, which is the ellipse with semi-major axis 2 and semi-minor axis 1.



Moreover, the Jacobian determinant is

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \left(\frac{\partial}{\partial u}2v\cos(u)\right)\left(\frac{\partial}{\partial v}v\sin(u)\right) - \left(\frac{\partial}{\partial v}2v\cos(u)\right)\left(\frac{\partial}{\partial u}v\sin(u)\right) \\ &= -2v\sin^2(u) - 2v\cos^2(u) \\ &= -2v\end{aligned}$$

Since S is type I, it follows that

$$\begin{aligned} Area\ of\ R &= \int\int_S \left| \frac{\partial (x,y)}{\partial (u,v)} \right| dA_{uv} \\ &= \int_0^{2\pi} \int_0^1 2v\ dvdu \\ &= \int_0^{2\pi} v^2 \Big|_0^1 du \\ &= \int_0^{2\pi} du \\ &= 2\pi \end{aligned}$$

Check your Reading: What is the image of $[0,\pi] \times [0,1]$ under

$$T(u,v) = \langle 2v\cos(u), v\sin(u) \rangle$$

Part 2: Change of Variable in Double Integrals

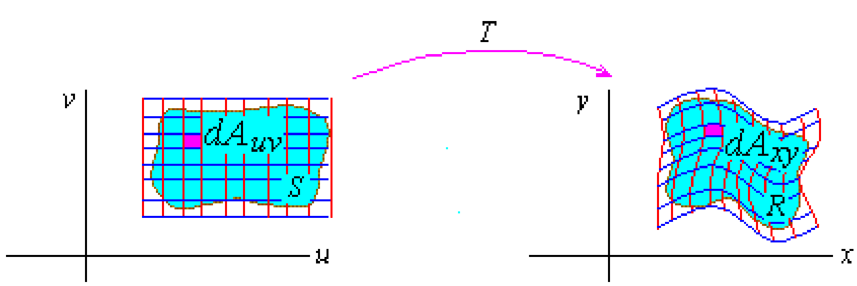
Let $\phi(x,y)$ be continuous on a region R that is the image under $T(u,v)$ of a region S in the uv-plane. Then the double integral over R is of the form

$$\int\int_R \phi(x,y)\, dA_{xy} = \lim_{h \to 0} \sum \sum \phi\left(x_i^*, y_j^*\right) \Delta A_{ij}$$

where $dA_{xy} = dA$ is the area differential in the xy-plane. If we choose $\left(u_i^*, v_j^*\right)$ such that $\left(x_i^*, y_j^*\right) = \left(f\left(u_i^*, v_j^*\right), g\left(u_i^*, v_j^*\right)\right)$, then similar to the discussion in part 1, we have

$$\begin{aligned} \int\int_R \phi(x,y)\, dA_{xy} &= \lim_{h\to 0} \sum\sum \phi\left(x_i^*, y_j^*\right) \Delta A_{ij} \\ &= \lim_{h\to 0} \sum\sum \phi\left(f\left(u_i^*, v_j^*\right), g\left(u_i^*, v_j^*\right)\right) \left| \frac{\partial (x,y)}{\partial (u,v)} \right|_{\left(u_i^*, v_j^*\right)} \Delta u_i \Delta v_j \\ &= \int\int_S \phi(f(u,v), g(u,v)) \left| \frac{\partial (x,y)}{\partial (u,v)} \right| dA_{uv} \end{aligned}$$

where dA_{uv} is the area differential on the uv-plane.



That is, is ϕ is continuous on R which is the image under $T(u,v) = \langle f(u,v), g(u,v) \rangle$ of a region S in the uv-plane, then

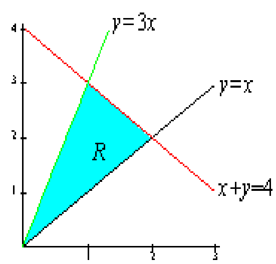
$$\int\int_R \phi(x,y)\, dA_{xy} = \int\int_S \phi(f(u,v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv} \tag{1}$$

The formula (1) is called the *change of variable* formula for double integrals, and the region S is called the *pullback* of R under T. In order to make the change of variables formula more usuable, let us notice that implementing (1) requires 3 steps:

1. (a) i. Compute the pullback S of R
   ii. Find the Jacobian and substitute for dA_{xy}
   iii. Replace x and y by $f(u,v)$ and $g(u,v)$, respectively.

Finally, evaluate the resulting double integral over S, using $dA_{uv} = dvdu$ if S is type I and $dA_{uv} = dudv$ if S is type II.

EXAMPLE 2 Evaluate $\int\int_R (x+y)\, dA$ where R is the region with boundaries $y = x$, $y = 3x$, and $x + y = 4$

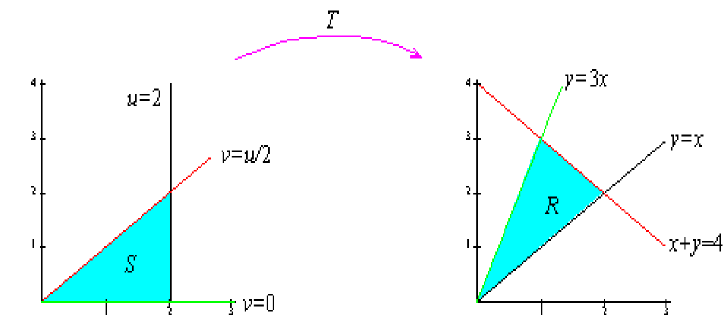


Use the transformation $T(u,v) = \langle u - v, u + v \rangle$.

Solution: To begin with, $T(u,v) = \langle u - v, u + v \rangle$ is equivalent to $x = u - v$, $y = u + v$. Thus, the pullback of the boundaries is as follows:

$$\begin{array}{rclcccccc} y & = & x & \Longrightarrow & u+v = u-v & \Longrightarrow & 2v = 0 & \Longrightarrow & v = 0 \\ y & = & 3x & \Longrightarrow & u+v = 3u-3v & \Longrightarrow & 4v = 2u & \Longrightarrow & v = \dfrac{u}{2} \\ x+y & = & 4 & \Longrightarrow & u-v+u+v = 4 & \Longrightarrow & 2u = 4 & \Longrightarrow & u = 2 \end{array}$$

That is, S is the region bounded by $v = 0$, $v = u/2$, $u = 2$.



The Jacobian of the transformation is

$$\frac{\partial\,(x,y)}{\partial\,(y,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 1 \cdot 1 - (-1) \cdot 1 = 2$$

Thus, $dA = 2dvdu$ and (1) implies that

$$\begin{aligned}\iint_R (x+y)\,dA_{xy} &= \iint_S (u - v + u + v)\,2dA_{uv} \\ &= \int_0^2 \int_0^{u/2} 4u\ dvdu \\ &= 16/3\end{aligned}$$

In example 2, we used the notation dA to state the problem and dA_{xy} in working the problem. This reflects the convention that if working solely within the xy-coordinate system, it is understood that $dA = dA_{xy}$.

Check your Reading: Is S in example 1 a type I or type II region?

Part 3: Converting Type I and Type II integrals

We can convert a type I or type II integral into different coordinates by first converting into a double integral and then using (1).

EXAMPLE 3 Use the transformation $T(u,v) = \langle u, u+v \rangle$ to evaluate the iterated integral

$$\int_1^2 \int_{x+2}^{x+3} \frac{dydx}{\sqrt{xy - x^2}}$$

Solution: To do so, we notice that as a double integral we have

$$\int_1^2 \int_{x+2}^{x+3} \frac{dydx}{\sqrt{xy - x^2}} = \int\int_R \frac{1}{\sqrt{xy - x^2}}\, dA$$

where R is the region bounded by $x = 1$, $x = 2$, $y = x + 2$ and $y = x + 3$. Since $T(u, v) = \langle u, u + v \rangle$ implies that $x = u$, $y = u + v$, we have

$$\begin{aligned}
y &= x + 2 \implies u + v = u + 2 \implies v = 2 \\
y &= x + 3 \implies u + v = u + 3 \implies v = 3 \\
x &= 1 \implies u = 1 \\
x &= 2 \implies u = 2
\end{aligned}$$

Thus, the pullback of R is $u = 1$, $u = 2$, $v = 2$, $v = 3$.

To compute the Jacobian, we notice that $x = u$, $y = u + v$ implies that

$$\frac{\partial (x, y)}{\partial (u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 1 \cdot 1 - 1 \cdot 0 = 1$$

Thus, $dA = 1dudv$, so that we transform the double integral into

$$\begin{aligned}
\int\int_R \frac{1}{\sqrt{xy - x^2}} dA &= \int\int_S \frac{1}{\sqrt{u(u + v) - u^2}} dA_{uv} \\
&= \int_1^2 \int_2^3 \frac{1}{\sqrt{u^2 + uv - u^2}} dudv \\
&= \int_1^2 \int_2^3 \frac{1}{\sqrt{uv}} dudv
\end{aligned}$$

As a result, we have

$$\begin{aligned}
\int_1^2 \int_{x+2}^{x+3} \frac{dydx}{\sqrt{xy - x^2}} &= \int_1^2 \int_2^3 u^{-1/2} v^{-1/2} dudv \\
&= 4\sqrt{2}\sqrt{3} - 8 - 4\sqrt{3} + 4\sqrt{2}
\end{aligned}$$

EXAMPLE 4 Use $T(u, v) = \left\langle u^2, v \right\rangle$ to evaluate

$$\int_0^1 \int_0^{\sqrt{x}} y e^{\sqrt{x}}\, dydx$$

Solution: The region R of integration is bounded the curves $x = 0$, $x = 1$, $y = 0$, and $y = \sqrt{x}$. Since $x = u^2$ and $y = v$, we have

$$\begin{aligned}
x &= 0 \implies u = 0 \\
x &= 1 \implies u = 1 \\
y &= 0 \implies v = 0 \\
y &= \sqrt{x} \implies v = u
\end{aligned}$$

Moreover, $x = u^2$ and $y = v$ implies that

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial}{\partial u}u^2\right)\left(\frac{\partial}{\partial v}v\right) - \left(\frac{\partial}{\partial v}u^2\right)\left(\frac{\partial}{\partial u}v\right) = 2u$$

Thus, $dA = 2ududv$, so that

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{x}} ye^{\sqrt{x}}\, dydx &= \int\int_R ye^{\sqrt{x}} dA_{xy} \\ &= \int\int_S ve^u \; 2u \; dA_{uv} \\ &= \int_0^1 \int_0^u ve^u \; 2u \; dvdu \\ &= \int_0^1 u^3 e^u du \\ &= -2e + 6 \end{aligned}$$

Check your Reading: What method was used in the final step of example 4?

Part 4: Change of Variable for Numerical Purposes

Many algorithms for approximating double integals numerically require the use of special types of regions and sophisticated methods for partitioning those regions. In such instances, the change of variables formula is often used as a tool for transforming a double integral into a form that is more amenable to numerical approximation, such as converting to integration over a rectangular region since a rectangle can be easily partitioned.

Most importantly, however, a change of variable might lead to the reduction of a double integral to a single integral, in which case only a single integral need be approximated numerically. In general, numerical methods for single integrals are preferable to numerical methods for multiple integrals.

EXAMPLE 5 Transform the following using $x = v\cosh(u)$, $y = v\sinh(u)$.

$$\int_0^{\sqrt{3}} \int_{2y}^{\sqrt{y^2+9}} \frac{\sin(x^2 - y^2)}{x^2 - y^2} dydx$$

Evaluate one of the integrals, and then approximate the remaining integral numberically.

Solution: To do so, we notice that as a double integral we have

$$\int_0^{\sqrt{3}} \int_{2y}^{\sqrt{y^2+9}} \frac{\sin\left(x^2-y^2\right)}{x^2-y^2} dxdy = \int\int_R \frac{\sin\left(x^2-y^2\right)}{x^2-y^2} dA_{xy}$$

where R is the region bounded by $y=0$, $x=2y$, and $x^2=y^2+9$.

picture

The pullback of R to S is accomplished by letting $x = v\cosh(u)$ and $y = v\sinh(u)$:

$$\begin{array}{rclclclcl} y &=& 0 &\Longrightarrow& v\sinh(u)=0 &\Longrightarrow& v=0 \textit{ or } \sinh(u)=0 &\Longrightarrow& v=0 \textit{ or } u=0 \\ x &=& 2y &\Longrightarrow& v\cosh(u)=2v\sinh(u) &\Longrightarrow& \tanh(u)=0.5 &\Longrightarrow& u=\tanh^{-1}(0.5) \\ x^2 &=& y^2+9 &\Longrightarrow& v^2\cosh^2(u)-v^2\sinh^2(u)=9 &\Longrightarrow& v^2=9 & & \end{array}$$

Thus, the pullback of R is $u=0$, $u=\tanh^{-1}(2)$, $v=0$, $v=3$.

We next compute the Jacobian determinant:

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= v\sinh(u)\sinh(u) - \cosh(u)\, v\cosh(u) \\ &= -v\left(\cosh^2(u)-\sinh^2(u)\right) \\ &= -v \end{aligned}$$

Thus, $dA_{xy} = |-v|\, dA_{uv} = v dA_{uv}$ since v is inside of S. Since $x^2 - y^2 = v^2$, the double integral becomes

$$\begin{aligned} \int\int_R \frac{\sin\left(x^2-y^2\right)}{x^2-y^2} dA_{xy} &= \int\int_S \frac{\sin\left(v^2\right)}{v^2}\ v\ dA_{uv} \\ &= \int_0^3 \int_0^{\tanh^{-1}(0.5)} \frac{\sin\left(v^2\right)}{v} dudv \\ &= \int_0^3 \frac{\sin\left(v^2\right)}{v} u\Big|_0^{\tanh^{-1}(0.5)} dv \\ &= \tanh^{-1}(0.5) \int_0^3 \frac{\sin\left(v^2\right)}{v} dv \end{aligned}$$

Finally, the remaining integral can be estimated numerically so that

$$\int_0^{\sqrt{3}} \int_{2y}^{\sqrt{y^2+9}} \frac{\sin\left(x^2-y^2\right)}{x^2-y^2} dydx = \tanh^{-1}(0.5) \int_0^3 \frac{\sin\left(v^2\right)}{v} dv \approx 0.4573$$

Exercises

Find and describe the image of the given region under the given transformation. Then compute the area of the image.

1. $T(u,v) = \langle 2u, 4v\rangle$
   $S = [0,1] \times [0,1]$
2. $T(u,v) = \langle u+3, 2v\rangle$
   $S = [0,1] \times [0,1]$
3. $T(u,v) = \langle u, 2u+v\rangle$
   $S = [0,1] \times [0,1]$
4. $T(u,v) = \langle u-v, u+v\rangle$
   $S = [0,1] \times [0,1]$
5. $T(u,v) = \langle u^2 - v^2, 2uv\rangle$
   $S = [0,1] \times [0,1]$
6. $T(u,v) = \langle u-v, uv\rangle$
   $S = [0,1] \times [0,1]$
7. $T(u,v) = \langle 4u\cos(v), 3u\sin(v)\rangle$,
   $S = [0,1] \times [0,2\pi]$
8. $T(u,v) = \langle u\cosh(v), u\sinh(v)\rangle$,
   $S = [0,1] \times [0,1]$

Use the given transformation to evaluate the given iterated integral.

9. $\int_0^1 \int_0^2 \frac{2xy}{x^2+1} dydx$
   $T(u,v) = \langle \sqrt{u}, v\rangle$
10. $\int_0^1 \int_0^1 xy\sin\left(y^2\right) dydx$
    $T(u,v) = \langle u, \sqrt{v}\rangle$
11. $\int_0^1 \int_0^1 e^x \cos(e^x)\, dxdy$
    $T(u,v) = \langle \ln(u), v\rangle$
12. $\int_0^1 \int_0^1 \cos(y)\, e^{\sin(y)} dydx$
    $T(u,v) = \left\langle u, \sin^{-1}(v)\right\rangle$
13. $\int_0^1 \int_0^x \cos\left(x^2\right) dydx$
    $T(u,v) = \langle v, u\rangle$
14. $\int_0^1 \int_x^1 \cos\left(y^2\right) dydx$
    $T(u,v) = \langle v, u\rangle$
15. $\int_1^2 \int_{2x}^{2x+1} \frac{1}{2y-4x} dydx$
    $T(u,v) = \langle u, 2u+v\rangle$
16. $\int_0^1 \int_{2x}^{2x+1} \sqrt{y-2x}\, dydx$
    $T(u,v) = \langle u, 2u+v\rangle$
17. $\int_0^1 \int_{y-1}^{y+1} \sin(y-x)\, dxdy$
    $T(u,v) = \langle u+v, v\rangle$
18. $\int_1^2 \int_y^{y+1} \frac{dxdy}{\sqrt{xy-y^2}}$
    $T(u,v) = \langle u+v, v\rangle$
19. $\int_0^1 \int_y^1 \sin\left(x^2\right) dxdy$
    $T(u,v) = \langle u, uv\rangle$
20. $\int_0^1 \int_x^{2x} e^{x^2} dydx$
    $T(u,v) = \langle u, uv\rangle$
21. $\int \int_R \sqrt{xy} dA$,
    $T(u,v) = \left\langle \frac{u}{v}, uv\right\rangle$
    R bounded by $xy = 1$, $xy = 9$
    $y = x$, $y = 4x$
22. $\int \int \sqrt{xy^3} dA$,
    $T(u,v) = \left\langle \frac{u}{v}, uv\right\rangle$
    R bounded by $xy = 1$, $xy = 9$
    $y = x$, $y = 4x$

Use the given transformation to transform the given iterated integral. Then *reduce to a single integral and approximate numerically.*

23. $\int_0^1 \int_{-x}^{x} \cos\left[(x-y)^2\right] dydx$
$T(u,v) = \langle v+u, v-u \rangle$

24. $\int_0^1 \int_{-x}^{x} \cos\left[(x+y)^2\right] dydx$
$T(u,v) = \langle v+u, v-u \rangle$

25. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} dydx$
$T(u,v) = \langle v\cos(u), v\sin(u) \rangle$

26. $\int_{-1}^1 \int_0^{0.5\sqrt{1-x^2}} \frac{\sin\left(x^2+4y^2\right)}{x^2+4y^2} dydx$
$T(u,v) = \langle 2v\cos(u), v\sin(u) \rangle$

27. $\int_0^2 \int_{y^2/4-1}^{1-y^2/4} \ln\left(16x^2 - 8xy^2 + y^4\right) dxdy$
$T(u,v) = \left\langle u^2 - v^2, 2uv \right\rangle$

28. $\int_0^2 \int_{y^2/4-1}^{1-y^2/4} e^{16x^2-8xy^2+y^4} dxdy$
$T(u,v) = \left\langle u^2 - v^2, 2uv \right\rangle$

29. Evaluate the following iterated integral in two ways:

$$\int_0^1 \int_0^3 2x\cos\left(x^2\right) dxdy$$

1. (a) By letting $w = x^2$, $dw = 2xdx$ and noting that $x = 0$ implies $w = 0$ while $x = 3$ implies $w = 9$.
   (b) Using the coordinate transformation

$$T(u,v) = \left\langle \sqrt{u}, v \right\rangle$$

30. Evaluate the following iterated integral in two ways:

$$\int_0^1 \int_0^4 \sqrt{x}\cos\left(\sqrt{x}\right) dxdy$$

1. (a) By letting $w = \sqrt{x}$, $dw = dx/\left(2\sqrt{x}\right)$ and noting that $x = 0$ implies $w = 0$ while $x = 4$ implies $w = 2$.
   (b) Using the coordinate transformation

$$T(u,v) = \left\langle u^2, v \right\rangle$$

31. In two different ways, we show that if g is a differentiable function, then

$$\int_a^b \int_{g(c)}^{g(d)} \phi(x,y)\, dydx = \int_a^b \int_c^d \phi(u, g(v))\, g'(v)\, dvdu$$

1. (a) By letting $y = g(v)$, $dy = g'(v)\, dv$ in the rightmost iterated integral.
   (b) By considering the coordinate transformation $T(u,v) = \langle u, g(v) \rangle$.

32. In two different ways, we show that if f is a differentiable function, then

$$\int_{f(a)}^{f(b)} \int_{g(c)}^{g(d)} \phi(x,y)\, dydx = \int_a^b \int_c^d \phi(f(u), g(v))\, f'(u)\, g'(v)\, dvdu$$

1. (a) By letting $x = f(u), dx = f'(u)\,du$ and $y = g(v)$, $dy = g'(v)\,dv$ in the rightmost iterated integral.

   (b) By considering the coordinate transformation $T(u,v) = \langle f(u), g(v) \rangle$

33. The *parabolic coordinate system* on the xy-plane is given by

$$T(u,v) = \left\langle u^2 - v^2, 2uv \right\rangle$$

If T maps a region S in the uv-plane to a region R in the xy-plane, then what does the change of variable formula imply that

$$\int\int_R \phi(x,y)\,dA$$

will become when integrated over S in the uv-coordinate system?

34. The *tangent coordinate system* on the xy-plane is given by

$$T(u,v) = \left\langle \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right\rangle$$

If T maps a region S in the uv-plane to a region R in the xy-plane, then what does the change of variable formula imply that

$$\int\int_R \phi(x,y)\,dA$$

will become when integrated over S in the uv-coordinate system?

35. The *elliptic coordinate system* on the xy-plane is given by

$$T(u,v) = \langle \cosh(u)\cos(v), \sinh(u)\sin(v) \rangle$$

If T maps a region S in the uv-plane to a region R in the xy-plane, then what does the change of variable formula imply that

$$\int\int_R \phi(x,y)\,dA$$

will become when integrated over S in the uv-coordinate system?

36. The *bipolar coordinate system* on the xy-plane is given by

$$T(u,v) = \left\langle \frac{\sinh(v)}{\cosh(v) - \cos(u)}, \frac{\sin(u)}{\cosh(v) - \cos(u)} \right\rangle$$

If T maps a region S in the uv-plane to a region R in the xy-plane, then what does the change of variable formula imply that

$$\int\int_R \phi(x,y)\,dA$$

will become when integrated over S in the uv-coordinate system?

37. Write to Learn: The *translation* $T(u,v) = \langle u+a, v+b \rangle$ translates a region S in the uv-plane to a region R in the xy-plane which is translated a units horizontally and b units vertically. Write a short essay which uses the change of coordinate formula to show that the area of R is the same as the area of S?

38. Write to Learn: A *rotation* $T(u,v) = \langle \cos(\theta)\,u + \sin(\theta)\,v, -\sin(\theta)\,u + \cos(\theta)\,v \rangle$ maps a region S in the uv-plane to a region R in the xy-plane which is rotated about the origin through an angle θ. Write a short essay which uses the change of coordinate formula to show that the area of R is the same as the area of S?