The Double Integral

Definition of the Integral

Iterated integrals are used primarily as a tool for computing double integrals, where a double integral is an integral of \( f(x, y) \) over a region \( R \). In this section, we define double integrals and begin examining how they are used in applications.

To begin with, a set of numbers \( \{x_0, x_j, r_j\}, \; j = 1, \ldots, m \), is said to be a tagged partition of \( [a, b] \) if

\[
a = x_0 < x_1 < x_2 < \ldots < x_m = b
\]

and if \( x_{j-1} \leq r_j \leq x_j \) for all \( j = 1, \ldots, m \). Moreover, if we let \( \Delta x_j = x_j - x_{j-1} \), then the partition is said to be \( h \)-fine if \( \Delta x_j \leq h \) for all \( j = 1, \ldots, n \).

If \( \{x_0, x_j, r_j\}, \; j = 1, \ldots, m \), is an \( h \)-fine tagged partition of \( [a, b] \), and if \( \{y_0, y_k, t_k\}, \; k = 1, \ldots, n \) is a \( l \)-fine tagged partition of \( [c, d] \), then the rectangles \( [x_{j-1}, x_j] \times [y_{k-1}, y_k] \) partition the rectangle \( [a, b] \times [c, d] \) and the points \( (r_j, t_k) \) are inside the rectangles \( [x_{j-1}, x_j] \times [y_{k-1}, y_k] \).

The Riemann sum of a function \( f(x, y) \) over this partition of \( [a, b] \times [c, d] \) is

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} f(r_j, t_k) \Delta x_j \Delta y_k
\]

We then define the double integral of \( f(x, y) \) over \( [a, b] \times [c, d] \) to be the limit as \( h, l \) approach 0 of Riemann sums over \( h, l \) fine partitions:

\[
\iiint_{[a,b] \times [c,d]} f(x, y) \, dA = \lim_{h \to 0} \lim_{l \to 0} \sum_{j=1}^{m} \sum_{k=1}^{n} f(s_j, t_k) \Delta x_j \Delta y_k
\]

To define the double integral over a bounded region \( R \) other than a rectangle,
we choose a rectangle \([a, b] \times [c, d]\) that contains \(R\),

\[
\begin{array}{c}
\hline
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
\]

\(d\)

\(c\)

\(R\)

\(a\)

\(b\)

and we define \(g\) so that \(g(x, y) = f(x, y)\) if \((x, y)\) is in \(R\) and \(g(x, y) = 0\) otherwise. The double integral of \(f(x, y)\) over an arbitrary region \(R\) is then defined to be

\[
\iint_{R} f(x, y) \, dA = \iint_{[a, b] \times [c, d]} g(x, y) \, dA
\]

It then follows from the definition that the double integral satisfies the following properties:

\[
\begin{align*}
\iint_{R} [f(x, y) + g(x, y)] \, dA &= \iint_{R} f(x, y) \, dA + \iint_{R} g(x, y) \, dA \quad (1) \\
\iint_{R} [f(x, y) - g(x, y)] \, dA &= \iint_{R} f(x, y) \, dA - \iint_{R} g(x, y) \, dA \quad (2) \\
\iint_{R} kf(x, y) \, dA &= k \iint_{R} f(x, y) \, dA \quad (3)
\end{align*}
\]

where \(k\) is a constant.

**EXAMPLE 1** Evaluate the integral of \(f + g\) over \(R\) if

\[
\iint_{R} f(x, y) = 3 \quad \text{and} \quad \iint_{R} g(x, y) = 2
\]  

(4)

**Solution:** We use property (1) to write

\[
\iint_{R} [f(x, y) + g(x, y)] \, dA = \iint_{R} f(x, y) \, dA + \iint_{R} g(x, y) \, dA = 3 + 2 = 5
\]

**Check your Reading:** What is the integral of \(f - g\) over \(R\) given (4)?
Volume

If \( f(x, y) \geq 0 \) on \([a, b] \times [c, d]\), then the \( f(r_j, t_k) \Delta x_j \Delta y_k \) is the volume of a "box" over a rectangle determined by the partitions of \([a, b]\) and \([c, d]\), respectively.

Consequently, the Riemann sum is an approximation of the volume of the solid
under $z = f(x, y)$ and over the rectangle $[a, b] \times [c, d]$.

Thus, if $f(x, y) \geq 0$ over $R$, then the volume of the solid below $z = f(x, y)$ and above $R$ is

$$V = \iiint_R f(x, y) \, dA$$

It follows from the previous section that if $R$ is a type I region bounded by $x = a, x = b, y = h(x), y = g(x)$, then

$$\iiint_R f(x, y) \, dA = \int_a^b \int_{h(x)}^{g(x)} f(x, y) \, dy \, dx$$

and if $R$ is a type II region bounded by $y = c, y = d, x = q(y), x = p(y)$, then

$$\iiint_R f(x, y) \, dA = \int_c^d \int_{q(y)}^{p(y)} f(x, y) \, dx \, dy$$

**EXAMPLE 2** Find the volume of the region below $z = x^2 y$ and over the region

$$R : \begin{array}{c} x = 0 \quad y = x \\ x = 1 \quad y = 1 \end{array}$$
Solution: Since the region is a type I region, we obtain
\[ V = \iint_R x^2 y \, dA = \int_0^1 \int_x^1 x^2 y \, dy \, dx \]
\[ = \int_0^1 x^2 \left[ y^2 \right]_x^1 \, dx \]
\[ = \int_0^1 \left( \frac{x^2}{2} - \frac{x^4}{2} \right) \, dx \]
\[ = \frac{1}{15} \]

In general, if \( f(x, y) \geq g(x, y) \) over a region \( R \),

then the volume of the solid between \( z = f(x, y) \) and \( z = g(x, y) \) over \( R \) is
\[ V = \iint_R [f(x, y) - g(x, y)] \, dA \quad (5) \]

If \( R \) is type I or type II, then (5) can be evaluated by reducing to either a type I or a type II integral, respectively.

**EXAMPLE 3**  Find the volume of the solid between \( z = x + y \) and \( z = x - y \) over the region
\[ R : \begin{align*}
y &= 0 & x &= y^2 \\
y &= 1 & x &= y \end{align*} \]

Solution: According to (5), the volume of the solid is
\[ V = \iint_R ((x + y) - (x - y)) \, dA = \iint_R 2y \, dA \]
which transforms into the type II iterated integral

\[ V = \int_0^1 \int_y^y 2y \, dx \, dy \]

Evaluating the inside integral results in

\[ V = \int_0^1 2yx \bigg|_y^y \, dy = \int_0^1 (2y \cdot y - 2y \cdot y^2) \, dy \]

It then follows that

\[ V = \int_0^1 (2y^2 - 2y^3) \, dy = \frac{1}{6} \]

**Check your Reading:** What type of region is the region \( R \) given in example 4?

### Converting Iterated Integrals into a Different Type

Many regions can be described as either type I or type II. As a result, a type I integral over such a region can be converted into a double integral, which can in turn be converted into a type II integral. This allows us to evaluate many iterated integrals that cannot be evaluated directly.

**EXAMPLE 4** Evaluate the iterated integral

\[ \int_0^1 \int_x^1 \sin (\pi y^2) \, dy \, dx \]  \( (6) \)

**Solution:** Since the antiderivative of \( \sin (y^2) \) cannot be expressed in closed form, the iterated integral (6) cannot be evaluated as a type I integral. Instead, we convert (6) to a double integral

\[ \int_0^1 \int_x^1 \sin (\pi y^2) \, dy \, dx = \iint_R \sin (\pi y^2) \, dA \]

and notice that the region \( R \) between \( x = 0, x = 1, y = x, \) and \( y = 1 \) can also be described as a type II region.
As a result, we can recast the original integral as a type II integral, thus leading to
\[
\int_0^1 \int_x^1 \sin (\pi y^2) \, dy \, dx = \int_R \sin (\pi y^2) \, dA = \int_0^1 \int_0^y \sin (\pi y^2) \, dx \, dy
\]
Not only did the description of the region change, but also the order of the differentials changed. Since \( \sin (y^2) \) is constant with respect to \( x \), we now have
\[
\int_0^1 \int_0^x \sin (\pi y^2) \, dx \, dy = \int_0^1 x \sin (\pi y^2) \bigg|_{0}^{y} \, dy = \int_0^1 y \sin (\pi y^2) \, dy
\]
The substitution \( u = y^2, \, du = 2y \, dy, \, u(0) = 0, \, u(1) = 1 \) then results in
\[
\int_0^1 \int_x^1 \sin (\pi y^2) \, dy \, dx = \int_0^1 \sin (\pi u) \, du = \frac{2}{\pi}
\]

**EXAMPLE 5** Evaluate the iterated integral
\[
\int_{-1}^1 \int_{|y|}^1 \sinh (y^3) \cos (x^2) \, dx \, dy \tag{7}
\]

**Solution:** The iterated integral (7) cannot be evaluated in closed form, so we instead convert (7) to a double integral:
\[
\int_0^1 \int_{|y|}^1 \sinh (y^3) \cos (x^2) \, dx \, dy = \int_R \sinh (y^3) \cos (x^2) \, dA
\]
The region \( R \) of integration is both type I and type II:

\[
\begin{array}{c|c|c}
\text{As a} & \text{Type II} & \text{As a} \\
y = -1 & x = |y| & x = 0 \\
y = 1 & x = 1 & y = -x \\
\end{array}
\]

Consequently, when transformed into a type I region we have
\[
\int_R \sinh (y^3) \cos (x^2) \, dA = \int_0^1 \int_{-x}^x \sinh (y^3) \cos (x^2) \, dy \, dx
\]
\[
= \int_0^1 \cos (x^2) \left[ \int_{-x}^x \sinh (y^3) \, dy \right] \, dx
\]
The resulting integral also cannot be evaluated in closed form, but because \( \sinh(y^3) \) is odd, we have
\[
\int_{-x}^{x} \sinh(y^3) \, dy = 0
\]
Thus, the entire integral must be zero, which means that
\[
\int_{-1}^{1} \int_{|y|}^{1} \sinh(y^3) \cos(x^2) \, dx \, dy = 0
\]

**Check your Reading:** Why can (7) not be evaluated in closed form?

**Fubini’s Theorem and Additional Properties**

The definition of the double integral implies many other properties. For example, if \( f(x, y) \leq g(x, y) \) on \( R \), then
\[
\iint_{R} f(x, y) \, dA \leq \iint_{R} g(x, y) \, dA
\]
and likewise, if \( f(x, y) \geq 0 \) on \( R \) and \( S \subset R \), then
\[
\iint_{S} f(x, y) \, dA \leq \iint_{R} f(x, y) \, dA
\]
Moreover, suppose that \( R \) and \( S \) are non-overlapping regions—i.e., that \( R \) and \( S \) do not intersect except possibly on the boundary:

Then as will be shown in the exercises, we must have
\[
\iint_{R \cup S} f(x, y) \, dA = \iint_{R} f(x, y) \, dA + \iint_{S} f(x, y) \, dA \tag{8}
\]
where \( R \cup S \) denotes the union of the regions \( R \) and \( S \).
Finally, properties of the double integral also follow from their relationship to iterated integrals. For example, since the rectangle \([a, b] \times [c, d]\) is both a type I and a type II region, we must have

\[
\iint_{[a,b] \times [c,d]} f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx \quad \text{and} \quad \iint_{[a,b] \times [c,d]} f(x,y) \, dA = \int_c^d \int_a^b f(x,y) \, dx \, dy
\]

As a result, the two iterated integrals are the same. This result is known as Fubini’s theorem, which says that if \(a, b, c\) and \(d\) are constant and if the double integral of \(f(x,y)\) exists, then

\[
\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy
\]

That is, the order of integration may be switched if the limits of integration are constant.

**EXAMPLE 6** Use Fubini’s theorem to evaluate

\[
\int_0^\pi \int_0^1 \cos(x) \sin(y^2) \, dy \, dx
\]

**Solution:** Fubini’s theorem implies that

\[
\int_0^\pi \int_0^1 \cos(x) \sin(y^2) \, dy \, dx = \int_0^1 \int_0^\pi \sin(y^2) \cos(x) \, dx \, dy
\]

As a result, we integrate \(\cos(x)\) to obtain

\[
\int_0^\pi \int_0^1 \cos(x) \sin(y^2) \, dy \, dx = \int_0^1 \sin(y^2) \sin(x) \bigg|_0^\pi \, dy = \int_0^1 \sin(y^2) (0 - 0) \, dy = 0
\]

**Exercises:**

*Find the volume of the solid between the graphs of the given functions over the given region:

1. \(f(x,y) = xy, \ g(x,y) = 0\)
   \(x = 0, \ x = 1, \ y = 0, \ y = 1\)

2. \(f(x,y) = x + 2y, \ g(x,y) = 0\)
   \(x = 1, \ x = 2, \ y = 0, \ y = 6\)

3. \(f(x,y) = x^2 + y^2, \ g(x,y) = 0\)
   \(y = 0, \ y = 1, \ x = y, \ x = 1\)

4. \(f(x,y) = x^3 + y^2, \ g(x,y) = 0\)
   \(y = 1, \ y = 2, \ x = y, \ x = y^2\)

5. \(f(x,y) = x + y, \ g(x,y) = x^2 + y^2\)
   \(x = 0, \ x = 1, \ y = 0, \ y = 1\)

6. \(f(x,y) = xy, \ g(x,y) = 4\)
   \(y = 0, \ y = 1, \ x = y, \ x = 1\)

7. \(f(x,y) = \sin(x), \ g(x,y) = 1,\)
   \(x = 0, \ x = \pi, \ y = 0, \ y = x\)

8. \(f(x,y) = \cos(x^2), \ g(x,y) = 1,\)
   \(x = 0, \ x = \pi, \ y = 0, \ y = x\)
Evaluate the iterated integral by changing it from type I to type II or vice versa:

9. \[ \int_0^1 \int_x^1 \cos (\pi y^2) \, dy \, dx \]
11. \[ \int_0^\pi \int_x^\pi \sin(y) \, dy \, dx \]
13. \[ \int_{-\pi/2}^{\pi/2} \int_{x \sin(y)}^{\pi/2-x} \, dx \, dy \]
15. \[ \int_1^4 \int_{x^2}^{\pi/2-x} \, dx \, dy \]

Evaluate using Fubini’s theorem.

17. \[ \int_0^1 \int_0^{2\pi} \sin(y) \, dx \, dy \]
19. \[ \int_0^1 \int_0^3 \cos(y) \, dx \, dy \]
21. \[ \int_{-\pi/2}^{\pi/2} \int_0^{\pi/2-x} \sin(x^2) \, dx \, dy \]
23. \[ \int_0^1 \int_0^1 e^{x+y} \, dx \, dy \]
25. \[ \int_0^1 \int_0^1 [f(x,y) + 2g(x,y)] \, dx \, dy \]
27. \[ \int_0^1 \int_0^1 [f(x,y) + g(x,y)] \, dx \, dy \]
29. \[ \int_0^1 \int_0^1 [f(x,y) + 2g(x,y)] \, dx \, dy \]

Use the properties of the double integrals and the double integrals

\[ \int_R f(x,y) \, dA = 5 \]
\[ \int_S f(x,y) \, dA = 7 \]
\[ \int_R g(x,y) \, dA = 11 \]

to evaluate the double integrals below:

24. \[ \int_0^1 \int_0^1 [f(x,y) - g(x,y)] \, dx \, dy \]
26. \[ \int_0^1 \int_0^1 [f(x,y) - 3g(x,y)] \, dx \, dy \]
28. \[ \int_0^1 \int_0^1 [f(x,y) + 3g(x,y)] \, dx \, dy \]
30. \[ \int_0^1 \int_0^1 [f(x,y) + g(x,y)] \, dx \, dy \]

31. Find the volume of the solid bound between the surfaces \( z = x^2 + y^2 \) and \( z = 9 \).
32. Find the volume of the solid bound between the surfaces \( z = x^2 + y^2 \) and \( z = 2x \) (hint: integrate over the region whose boundary curve is the intersection of the two surfaces).
33. Show that for all \((x,y)\) in \([0,1] \times [0,1]\) that

\[ 0 \leq \frac{\sin(\pi x)}{1 + \cos^2(y)} \leq \sin(\pi x) \]

and then use this result to estimate

\[ \int_0^1 \int_0^1 \frac{\sin(\pi x)}{1 + \cos^2(y)} \, dy \, dx \]

34. Let \( D \) denote the unit circle. Explain why

\[ \int_D e^{x+y} \, dA \leq \int_{-1}^1 \int_{-1}^1 e^{x+y} \, dy \, dx \]

and then evaluate this last integral.
35. Suppose that \( f(x) \geq 0 \) over \([a, b]\) and recall that the surface of revolution obtained by revolving the graph of \( f \) about the \( x \)-axis is given by

\[
r(u, v) = \langle v, f(v) \cos(u), f(v) \sin(u) \rangle
\]

for \( u \) in \([0, 2\pi]\) and \( v \) in \([a, b]\). Show that the volume of the resulting solid of revolution is

\[
\int_a^b \int_{-f(x)}^{f(x)} \sqrt{[f(x)]^2 - y^2} \, dy \, dx
\]

and then compute the innermost integral using the trigonometric substitution

\[
y = f(x) \sin(\theta)
\]

36. Suppose that \( f(x) > 0 \) for all \( x \) in \((a, b)\) and suppose that \( f(a) = f(b) = 0 \). What is the volume of the solid enclosed by the surface

\[
y^2 + z^2 = [f(x)]^2
\]

37. Use the Riemann definition of the double integral to prove (3).

38. Use the Riemann definition of the double integral to prove (1).

39. **Write to Learn:** Suppose that \( f(x, y) \) is integrable over two bounded, non-overlapping regions \( R \) and \( S \). Let \( g_1(x, y) = f(x, y) \) if \( (x, y) \) is in \( R \) and \( g_1(x, y) = 0 \) if \( (x, y) \) is not in \( R \). Similarly, let \( g_2(x, y) = f(x, y) \) if \( (x, y) \) is in \( S \) and let \( g_2(x, y) = 0 \) otherwise. Write a short essay in which you show that

\[
\iint_{R \cup S} f(x, y) \, dA = \iint_{[a, b] \times [c, d]} [g_1(x, y) + g_2(x, y)] \, dA
\]

where \([a, b] \times [c, d]\) contains \( R \cup S \). Then in that essay use this result to prove (8).

40. **Write to Learn:** Write a short essay in which you show that

\[
\int_a^b \int_c^d f(x) g(y) \, dy \, dx = \left[ \int_c^d f(x) \, dx \right] \left[ \int_a^b g(y) \, dy \right]
\]