The Double Integral

Definition of the Integral

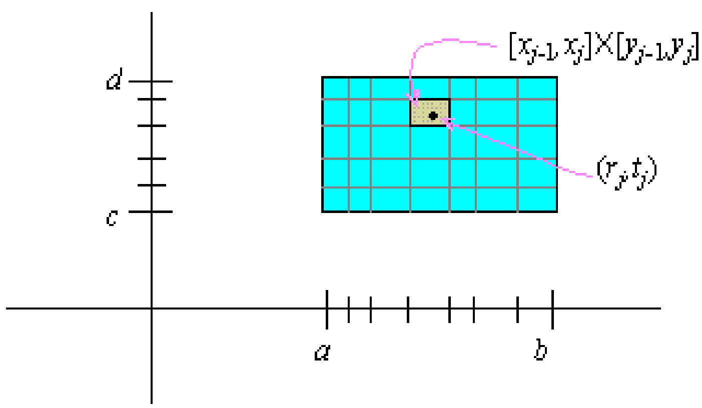
Iterated integrals are used primarily as a tool for computing *double integrals,* where a double integral is an integral of $f(x,y)$ over a region R. In this section, we define double integrals and begin examining how they are used in applications.

To begin with, a set of numbers $\{x_0, x_j, r_j\}$, $j = 1, \ldots, m$, is said to be a *tagged partition* of $[a,b]$ if

$$a = x_0 < x_1 < x_2 < \ldots < x_m = b$$

and if $x_{j-1} \leq r_j \leq x_j$ for all $j = 1, ..., m$. Moreover, if we let $\Delta x_j = x_j - x_{j-1}$, then the partition is said to be h-fine if $\Delta x_j \leq h$ for all $j = 1, \ldots, n$.

If $\{x_0, x_j, r_j\}$, $j = 1, \ldots, m$, is an h-fine tagged partition of $[a,b]$, and if $\{y_0, y_k, t_k\}$, $k = 1, \ldots, n$ is a l-fine tagged partition of $[c,d]$, then the rectangles $[x_{j-1}, x_j] \times [y_{k-1}, y_k]$ partition the rectangle $[a,b] \times [c,d]$ and the points (r_j, t_k) are inside the rectangles $[x_{j-1}, x_j] \times [y_{k-1}, y_k]$.



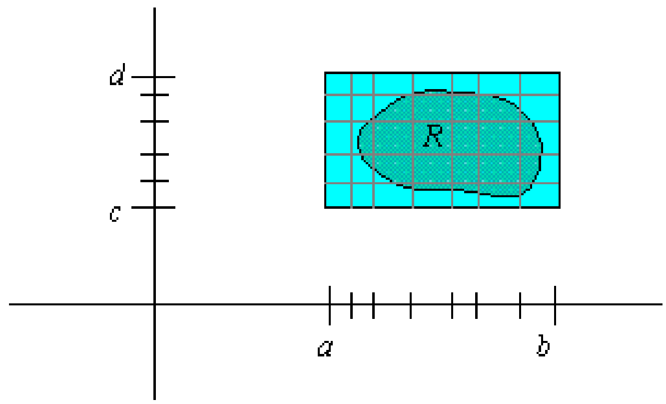
The *Riemann sum* of a function $f(x,y)$ over this partition of $[a,b] \times [c,d]$ is

$$\sum_{j=1}^{m} \sum_{k=1}^{n} f(r_j, t_k)\, \Delta x_j \Delta y_k$$

We then define the *double integral* of $f(x,y)$ over $[a,b] \times [c,d]$ to be the limit as h, l approach 0 of Riemann sums over h, l fine partitions:

$$\iint_{[a,b]\times[c,d]} f(x,y)\, dA = \lim_{h\to 0} \lim_{l\to 0} \sum_{j=1}^{m} \sum_{k=1}^{n} f(s_j, t_k)\, \Delta x_j \Delta y_k$$

To define the double integral over a bounded region R other than a rectangle, we choose a rectangle $[a,b] \times [c,d]$ that contains R,



and we define g so that $g(x,y) = f(x,y)$ if (x,y) is in R and $g(x,y) = 0$ otherwise. The double integral of $f(x,y)$ over an arbitrary region R is then defined to be

$$\iint_R f(x,y)\,dA = \iint_{[a,b]\times[c,d]} g(x,y)\,dA$$

It then follows from the definition that the double integral satisfies the following properties:

$$\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA \qquad (1)$$

$$\iint_R [f(x,y) - g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA \qquad (2)$$

$$\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA \qquad (3)$$

where k is a constant.

EXAMPLE 1 Evaluate the integral of $f+g$ over R if

$$\iint_R f(x,y) = 3 \quad \textit{and} \quad \iint_R g(x,y) = 2 \qquad (4)$$

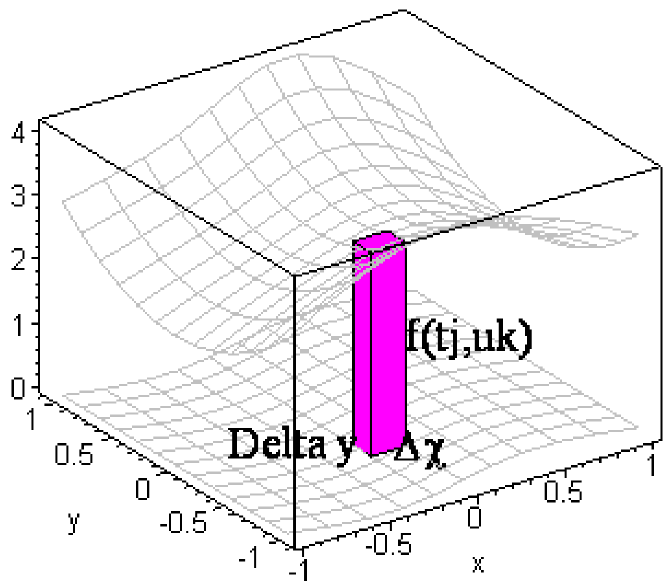
Solution: We use property (1) to write

$$\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA = 3+2 = 5$$

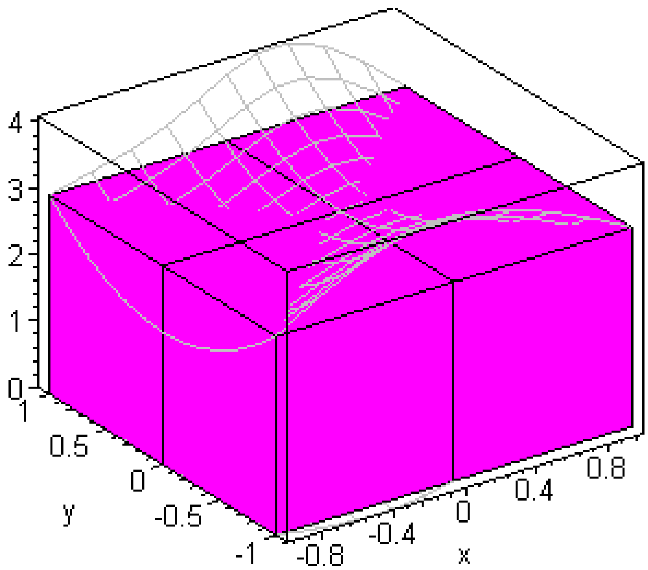
Check your Reading: What is the integral of $f-g$ over R given (4)?

Volume

If $f(x,y) \geq 0$ on $[a,b] \times [c,d]$, then the $f(r_j, t_k)\, \Delta x_j \Delta y_k$ is the volume of a "box" over a rectangle determined by the partitions of $[a,b]$ and $[c,d]$, respectively.



Consequently, the Riemann sum is an approximation of the volume of the solid under $z = f(x, y)$ and over the rectangle $[a, b] \times [c, d]$.



Thus, if $f(x, y) \geq 0$ over R, then the volume of the solid below $z = f(x, y)$ and above R is

$$V = \iint_R f(x, y)\, dA$$

It follows from the previous section that if R is a type I region bounded by $x = a$, $x = b$, $y = h(x)$, $y = g(x)$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{h(x)}^{g(x)} f(x, y)\, dydx$$

and if R is a type II region bounded by $y = c$, $y = d$, $x = q(y)$, $x = p(y)$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{q(y)}^{p(y)} f(x, y)\, dxdy$$

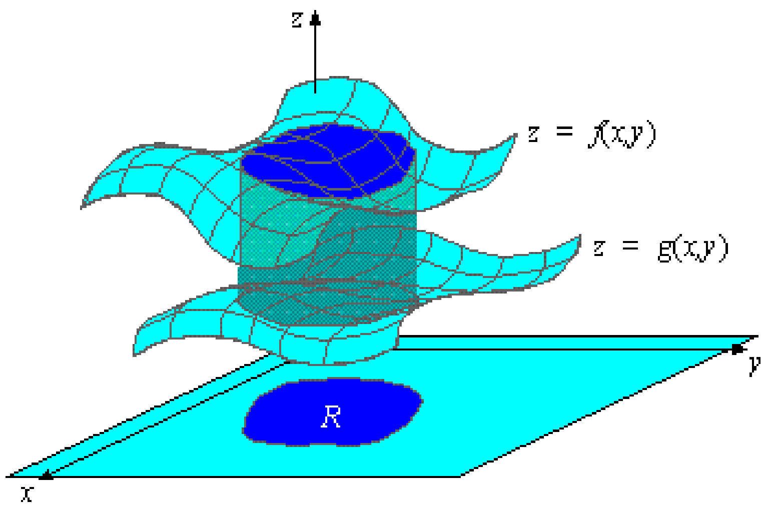
EXAMPLE 2 Find the volume of the region below $z = x^2 y$ and over the region

$$R: \quad \begin{array}{ll} x = 0 & y = x \\ x = 1 & y = 1 \end{array}$$

Solution: Since the region is a type I region, we obtain

$$\begin{aligned} V &= \iint_R x^2 y \, dA = \int_0^1 \int_x^1 x^2 y \, dy dx \\ &= \int_0^1 \left. \frac{x^2 y^2}{2} \right|_x^1 dx \\ &= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx \\ &= \frac{1}{15} \end{aligned}$$

In general, if $f(x, y) \geq g(x, y)$ over a region R,



then the volume of the solid between $z = f(x, y)$ and $z = g(x, y)$ over R is

$$V = \iint_R [f(x, y) - g(x, y)] \, dA \tag{5}$$

If R is type I or type II, then (5) can be evaluated by reducing to either a type I or a type II integral, respectively.

EXAMPLE 3 Find the volume of the solid between $z = x + y$ and $z = x - y$ over the region

$$R: \quad \begin{array}{ll} y = 0 & x = y^2 \\ y = 1 & x = y \end{array}$$

Solution: According to (5), the volume of the solid is

$$V = \iint_R ((x + y) - (x - y)) \, dA = \iint_R 2y \, dA$$

which transforms into the type II iterated integral

$$V = \int_0^1 \int_{y^2}^{y} 2y \; dxdy$$

Evaluating the inside integral results in

$$V = \int_0^1 2yx|_{y^2}^{y} \, dy = \int_0^1 \left(2y \cdot y - 2y \cdot y^2\right) dy$$

It then follows that

$$V = \int_0^1 \left(2y^2 - 2y^3\right) dy = \frac{1}{6}$$

Check your Reading: What type of region is the region R given in example 4?

Converting Iterated Integrals into a Different Type

Many regions can be described as either type I or type II. As a result, a type I integral over such a region can be converted into a double integral, which can in turn be converted into a type II integral. This allows us to evaluate many iterated integrals that cannot be evaluated directly.

EXAMPLE 4 Evaluate the iterated integral

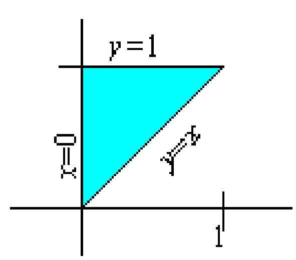
$$\int_0^1 \int_x^1 \sin\left(\pi y^2\right) dydx \tag{6}$$

Solution: Since the antiderivative of $\sin\left(y^2\right)$ cannot be expressed in closed form, the iterated integral (6) cannot be evaluated as a type I integral. Instead, we convert (6) to a double integral

$$\int_0^1 \int_x^1 \sin\left(\pi y^2\right) dydx = \iint_R \sin\left(\pi y^2\right) dA$$

and notice that the region R between $x = 0$, $x = 1$, $y = x$, and $y = 1$ can also be described as a type II region.

As a	Type I
$x = 0$	$y = x$
$x = 1$	$y = 1$



As a	Type II
$y = 0$	$x = 0$
$y = 1$	$x = y$

As a result, we can recast the original integral as a type II integral, thus leading to

$$\int_0^1 \int_x^1 \sin\left(\pi y^2\right) dydx = \iint_R \sin\left(\pi y^2\right) dA = \int_0^1 \int_0^y \sin\left(\pi y^2\right) dxdy$$

Not only did the description of the region change, but also the order of the differentials changed. Since $\sin\left(y^2\right)$ is constant with respect to x, we now have

$$\begin{aligned} \int_0^1 \int_0^y \sin\left(\pi y^2\right) dxdy &= \int_0^1 x \sin\left(\pi y^2\right)\Big|_0^y dy \\ &= \int_0^1 y \sin\left(\pi y^2\right) dy \end{aligned}$$

The substitution $u = y^2$, $du = 2ydy$, $u(0) = 0$, $u(1) = 1$ then results in

$$\int_0^1 \int_x^1 \sin\left(\pi y^2\right) dydx = \int_0^1 \sin(\pi u)\, du = \frac{2}{\pi}$$

EXAMPLE 5 Evaluate the iterated integral

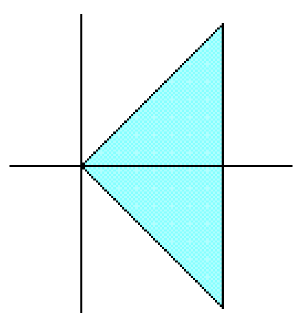
$$\int_{-1}^1 \int_{|y|}^1 \sinh\left(y^3\right) \cos\left(x^2\right)\ dxdy \qquad (7)$$

Solution: The iterated integral (7) cannot be evaluated in closed form, so we instead convert (7) to a double integral:

$$\int_0^1 \int_{|y|}^1 \sinh\left(y^3\right) \cos\left(x^2\right)\ dxdy = \iint_R \sinh\left(y^3\right) \cos\left(x^2\right)\ dA$$

The region R of integration is both type I and type II:

As a	Type II
$y = -1$	$x = \lvert y \rvert$
$y = 1$	$x = 1$



As a	Type I
$x = 0$	$y = -x$
$x = 1$	$y = x$

Consequently, when transformed into a type I region we have

$$\begin{aligned} \iint_R \sinh\left(y^3\right) \cos\left(x^2\right)\ dA &= \int_0^1 \int_{-x}^x \sinh\left(y^3\right) \cos\left(x^2\right) dydx \\ &= \int_0^1 \cos\left(x^2\right) \left[\int_{-x}^x \sinh\left(y^3\right) dy\right] dx \end{aligned}$$

The resulting integral also cannot be evaluated in closed form, but because $\sinh\left(y^3\right)$ is odd, we have

$$\int_{-x}^{x} \sinh\left(y^3\right) dy = 0$$

Thus, the entire integral must be zero, which means that

$$\int_{-1}^{1}\int_{|y|}^{1} \sinh\left(y^3\right)\cos\left(x^2\right)\ dxdy = 0$$

Check your Reading: Why can (7) not be evaluated in closed form?

Fubini's Theorem and Additional Properties

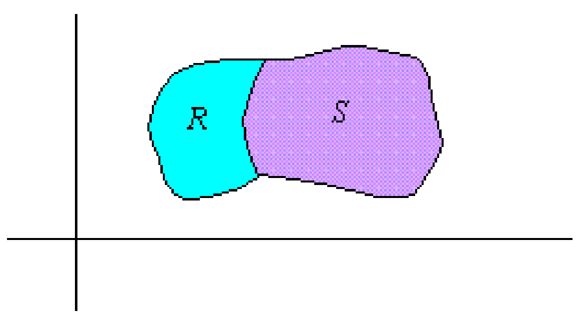
The definition of the double integral implies many other properties. For example, if $f(x,y) \leq g(x,y)$ on R, then

$$\iint_R f(x,y)\,dA \leq \iint_R g(x,y)\,dA$$

and likewise, if $f(x,y) \geq 0$ on R and $S \subset R$, then

$$\iint_S f(x,y)\,dA \leq \iint_R f(x,y)\,dA$$

Moreover, suppose that R and S are non-overlapping regions–i.e., that R and S do not intersect except possibly on the boundary:



Then as will be shown in the exercises, we must have

$$\iint_{R\cup S} f(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_S f(x,y)\,dA \qquad (8)$$

where $R \cup S$ denotes the *union* of the regions R and S.

Finally, properties of the double integral also follow from their relationship to iterated integrals.. For example, since the rectangle $[a,b]\times[c,d]$ is both a type I and a type II region, we must have

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dydx \qquad \textit{and} \qquad \iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy$$

As a result, the two iterated integrals are the same. This result is known as *Fubini's theorem,* which says that if a,b,c and d are constant and if the double integral of $f(x,y)$ exists, then

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy$$

That is, the order of integration may be switched if the limits of integration are constant.

EXAMPLE 6 Use Fubini's theorem to evaluate

$$\int_0^\pi \int_0^1 \cos(x)\sin\left(y^2\right)dydx$$

Solution: Fubini's theorem implies that

$$\int_0^\pi \int_0^1 \cos(x)\sin\left(y^2\right)dydx = \int_0^1 \int_0^\pi \sin\left(y^2\right)\cos(x)\,dxdy$$

As a result, we integrate $\cos(x)$ to obtain

$$\begin{aligned}\int_0^\pi \int_0^1 \cos(x)\sin\left(y^2\right)dydx &= \int_0^1 \sin\left(y^2\right)\left.\sin(x)\right|_0^\pi\,dy \\ &= \int_0^1 \sin\left(y^2\right)(0-0)\,dy \\ &= 0\end{aligned}$$

Exercises:

Find the volume of the solid between the graphs of the given functions over the given region:

1. $f(x,y) = xy,\ g(x,y) = 0$
   $x=0,\ x=1,\ y=0,\ y=1$
2. $f(x,y) = x+2y,\ g(x,y) = 0$
   $x=1,\ x=2,\ y=0,\ y=6$
3. $f(x,y) = x^2+y^2,\ g(x,y) = 0$
   $y=0,\ y=1,\ x=y,\ x=1$
4. $f(x,y) = x^3+y^2,\ g(x,y) = 0$
   $y=1,\ y=2,\ x=y,\ x=y^2$
5. $f(x,y) = x+y,\ g(x,y) = x^2+y^2$
   $x=0,\ x=1,\ y=0,\ y=1$
6. $f(x,y) = xy,\ g(x,y) = 4$
   $y=0,\ y=1,\ x=y,\ x=1$
7. $f(x,y) = \sin(x),\ g(x,y) = 1,$
   $x=0,\ x=\pi,\ y=0,\ y=x$
8. $f(x,y) = \cos\left(x^2\right),\ g(x,y) = 1,$
   $x=0,\ x=\pi,\ y=0,\ y=x$

Evaluate the iterated integral by changing it from type I to type II or vice versa:

9.	$\int_0^1 \int_x^1 \cos\left(\pi y^2\right) dydx$	10.	$\int_0^1 \int_y^1 2y \sin\left(\pi x^3\right) dxdy$
11.	$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} dydx$	12.	$\int_{-1}^1 \int_{\lvert y\rvert}^1 \sin\left(x^2 y^3\right) dxdy$
13.	$\int_0^1 \int_{\sin^{-1}(x)}^{\pi/2} x \csc(y)\, dydx$	14.	$\int_0^2 \int_{x^2}^4 e^{x/\sqrt{y}} dydx$
15.	$\int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y} dxdy$	16.	$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx$

Evaluate using Fubini's theorem.

17.	$\int_0^1 \int_0^{2\pi} x \sin(y)\ dydx$	18.	$\int_{-1}^1 \int_0^3 x \sin\left(y^2\right) dxdy$
19.	$\int_0^1 \int_0^3 e^{xy}\, dydx$	20.	$\int_{-1}^1 \int_0^3 \sinh(xy)\, dxdy$
21.	$\int_{-\pi}^{\pi} \int_0^{\pi}\ \sin\left(x^2 y\right) dxdy$	22.	$\int_{-\pi/4}^{\pi/4} \int_0^{\pi}\ \tan^2(y) \tan(x)\, dxdy$

Use the properties of the double integrals and the double integrals

$$\iint_R f(x,y)\, dA = 5 \qquad \iint_S f(x,y)\, dA = 7 \qquad \iint_R g(x,y)\, dA = 11$$

to evaluate the double integrals below:

23.	$\iint_R 7f(x,y)\, dA$	24.	$\iint_R [f(x,y) - g(x,y)]\, dA$
25.	$\iint_R [f(x,y) + 2g(x,y)]\, dA$	26.	$\iint_R [f(x,y) - 3f(x,y)]\, dA$
27.	$\iint_{R\cup S} f(x,y)\, dA$	28.	$\iint_{R\cup S} 7f(x,y)\, dA$
29.	$\iint_{R\cup S} g(x,y)\, dA - \iint_S g(x,y)\, dA$	30.	$\iint_{R\cup S} [f(x,y) + g(x,y)]\, dA - \iint_S g(x,y)\, dA$

31. Find the volume of the solid bound between the surfaces $z = x^2 + y^2$ and $z = 9$.

32. Find the volume of the solid bound between the surfaces $z = x^2+y^2$ and $z = 2x$. (hint: integrate over the region whose boundary curve is the intersection of the two surfaces).

33. Show that for all (x, y) in $[0,1] \times [0,1]$ that

$$0 \le \frac{\sin(\pi x)}{1 + \cos^2(y)} \le \sin(\pi x)$$

and then use this result to estimate

$$\int_0^1 \int_0^1 \frac{\sin(\pi x)}{1 + \cos^2(y)} dydx$$

34. Let **D** denote the unit circle. Explain why

$$\iint_D e^{x+y} dA \le \int_{-1}^1 \int_{-1}^1 e^{x+y} dydx$$

and then evaluate this last integral.

35. Suppose that $f(x) \geq 0$ over $[a,b]$ and recall that the surface of revolution obtained by revolving the graph of f about the x-axis is given by

$$\mathbf{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u)\rangle$$

for u in $[0,2\pi]$ and v in $[a,b]$. Show that the volume of the resulting solid of revolution is

$$\int_a^b \int_{-f(x)}^{f(x)} \sqrt{[f(x)]^2 - y^2}\,dy\,dx$$

and then compute the innermost integral using the trigonometric substitution

$$y = f(x)\sin(\theta)$$

36. Suppose that $f(x) > 0$ for all x in (a,b) and suppose that $f(a) = f(b) = 0$. What is the volume of the solid enclosed by the surface

$$y^2 + z^2 = [f(x)]^2$$

37. Use the Riemann definition of the double integral to prove (3).

38. Use the Riemann definition of the double integral to prove (1).

39. Write to Learn: Suppose that $f(x,y)$ is integrable over two bounded, non-overlapping regions R and S. Let $g_1(x,y) = f(x,y)$ if (x,y) is in R and $g_1(x,y) = 0$ if (x,y) is not in R. Similarly, let $g_2(x,y) = f(x,y)$ if (x,y) is in S and let $g_2(x,y) = 0$ otherwise. Write a short essay in which you show that

$$\iint_{R\cup S} f(x,y)\,dA = \iint_{[a,b]\times[c,d]} [g_1(x,y) + g_2(x,y)]\,dA$$

where $[a,b]\times[c,d]$ contains $R\cup S$. Then in that essay use this result to prove (8).

40. Write to Learn: Write a short essay in which you show that

$$\int_a^b \int_c^d f(x)\,g(y)\,dx\,dy = \left[\int_c^d f(x)\,dx\right]\left[\int_a^b g(y)\,dy\right]$$