Iterated Integrals

Type I Integrals

In this section, we begin the study of integrals over regions in the plane. To do so, however, requires that we examine the important idea of *iterated integrals*, in which indefinite integrals are the integrand of a definite integral.

To begin with, we define a *type I* iterated integral to be an integral of the form

$$\int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx$$

To evaluate a type I integral, we first evaluate the inner integral

$$\int_{p(x)}^{q(x)} f(x,y)\,dy$$

treating x as a constant. We then evaluate the result with respect to x:

$$\int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{p(x)}^{q(x)} f(x,y)\,dy \right] dx$$

EXAMPLE 1 Evaluate the type I integral

$$\int_0^1 \int_0^x \left(xy^2+1\right) dydx$$

Solution: To begin with, we integrate with respect to y:

$$\begin{aligned} \int_0^x \left(xy^2+1\right) dy &= \left.\left(x\frac{y^3}{3}+y\right)\right|_0^x \\ &= \left(x\frac{x^3}{3}+x\right) - \left(x\frac{0^3}{3}+0\right) \\ &= \frac{1}{3}x^4+x \end{aligned}$$

As a result, we have

$$\begin{aligned} \int_0^1 \int_0^x \left(xy^2+1\right) dydx &= \int_0^1 \left(\frac{1}{3}x^4+x\right) dx \\ &= \left.\frac{1}{3}\frac{x^4}{4}+\frac{x^2}{2}\right|_0^1 \\ &= \frac{1}{12}+\frac{1}{2} \\ &= \frac{7}{12} \end{aligned}$$

Often we evaluate the innermost integral inside the integrand of the outer integral rather than writing the integrations separately.

EXAMPLE 2 Evaluate the type I integral

$$\int_0^2 \int_1^x x^2 y dy dx$$

Solution: We first evaluate the inner integral:

$$\begin{aligned}
\int_0^2 \int_1^x x^2 y dy dx &= \int_0^2 \left[\int_1^x x^2 y dy \right] dx \\
&= \int_0^2 \left[x^2 \frac{y^2}{2} \Big|_1^x \right] dx \\
&= \int_0^2 \left[x^2 \frac{x^2}{2} - x^2 \frac{1}{2} \right] dx \\
&= \int_0^2 \left[\frac{x^4}{2} - \frac{x^2}{2} \right] dx \\
&= \frac{28}{15}
\end{aligned}$$

Check your Reading: Why is 15 the denominator of the result in example 2?

Type II Integrals

Similarly, we define a *type II integral* to be an iterated integral of the form

$$\int_c^d \int_{u(y)}^{v(y)} f(x, y) \, dx dy$$

It is evaluated by considering y to be constant in the innermost integral, and then integrating the result with respect to y.

EXAMPLE 3 Evaluate the type II integral

$$\int_0^1 \int_{y^2}^{y} (x + y) \, dx dy$$

Solution: We treat y as a constant in the innermost integral:

$$\begin{aligned}
\int_0^1 \int_{y^2}^{y} (x-y)\,dxdy &= \int_0^1 \left[\int_{y^2}^{y} (2x-y)\,dx\right] dy \\
&= \int_0^1 \left[x^2 - xy\big|_{y^2}^{y}\right] dy \\
&= \int_0^1 \left[\left(y^2-y^2\right) - \left(\left(y^2\right)^2 - y^2 y\right)\right] dy \\
&= \int_0^1 \left(y^4 - y^3\right) dy \\
&= \frac{-1}{20}
\end{aligned}$$

EXAMPLE 4 Evaluate the type II integral

$$\int_0^{\pi} \int_0^{y} \sin(y)\,dxdy$$

Solution: Since we treat y as a constant in the innermost integral, the function $\sin(y)$ can be considered constant and

$$\begin{aligned}
\int_0^{\pi} \int_0^{y} \sin(y)\,dxdy &= \int_0^{\pi} \left[\sin(y) \int_0^{y} dx\right] dy \\
&= \int_0^{\pi} [y \sin(y)]\,dy
\end{aligned}$$

We now use integration by parts with $u = y$ and $dv = \sin(y)\,dy$ to obtain

$$\begin{array}{cc} u = y & dv = \sin(y)\,dy \\ du = dy & v = -\cos(y) \end{array} \qquad \int_0^{\pi} [y\sin(y)]\,dy = -y\cos(y)\big|_0^{\pi} + \int_0^{\pi} \cos(y)\,dy$$

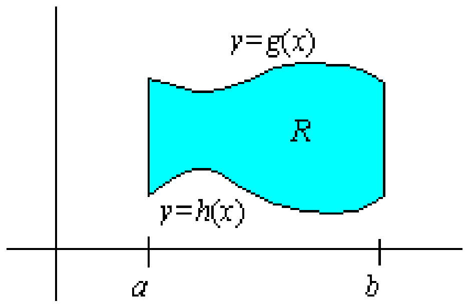
As a result, we have

$$\int_0^{\pi} \int_0^{y} \sin(y)\,dxdy = -\pi\cos(\pi) = \pi$$

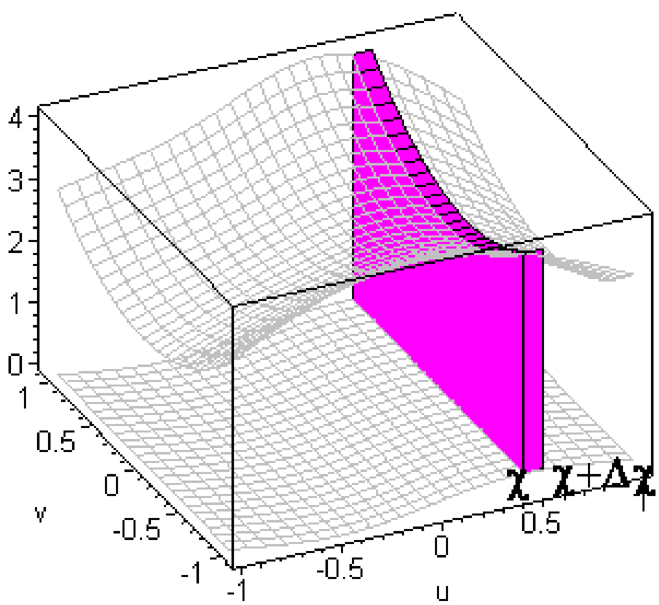
Check your Reading: Why can we write $\int_0^y \sin(y)\,dx$ as $\sin(y)\int_0^y dx$?

Volumes of Solids over Type I Regions

Let g, h be continuous on $[a, b]$ and supppose that $g(x) \leq h(x)$ for x in $[a, b]$. If R is a region in the xy-plane which is bounded by the curves $x = a$, $x = b$, $y = g(x)$, and $y = h(x)$,

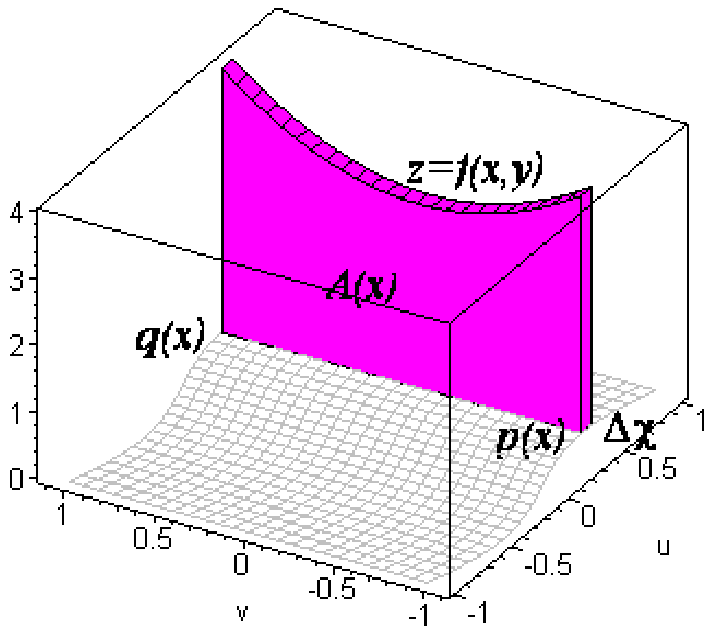


then R is said to be a *type I region*. Let's find the volume of the solid between the graph of $f(x, y)$ and the xy-plane over a type I region R when $f(x, y) \geq 0$.



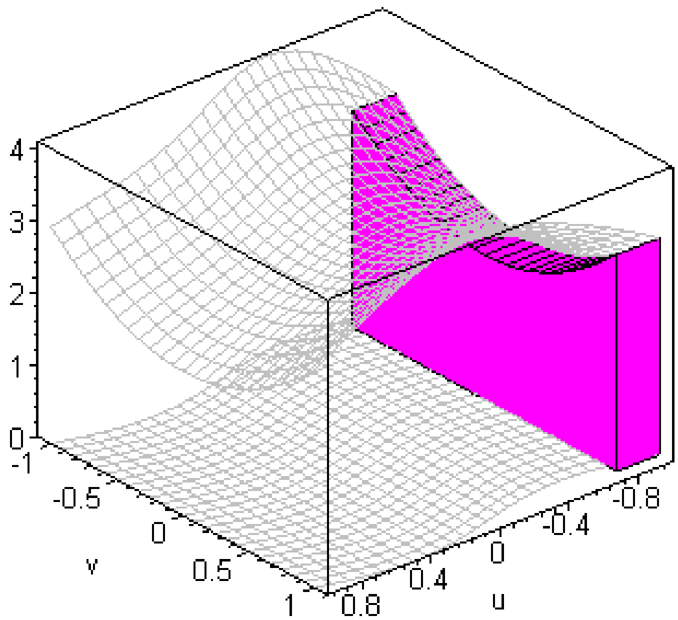
To do so, let's notice that if the solid is sliced with a plane parallel to the xz-plane, then its area is

$$A(x) = \int_{g(x)}^{h(x)} f(x, y)\, dy$$



It follows that if $\{x_j, t_j\}$, $j = 1, \ldots, n$, is a tagged partition of $[a, b]$, then the volume of the solid under the graph of $f(x, y)$ and over the region R is

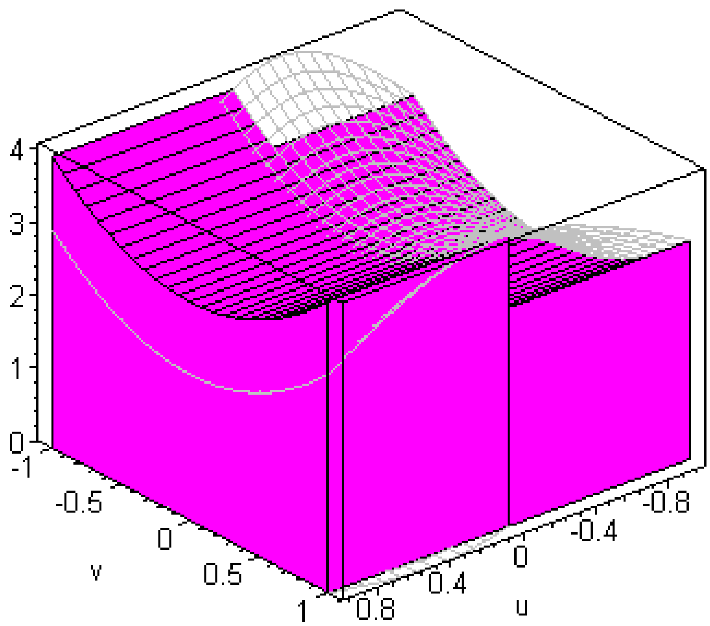
$$V \approx \sum_{j=1}^{n} A(t_j) \Delta x_j$$



A limit of such simple function approximations yields the *volumes by slicing* formula

$$V = \int_a^b A(x)\, dx$$

which is illustrated below:

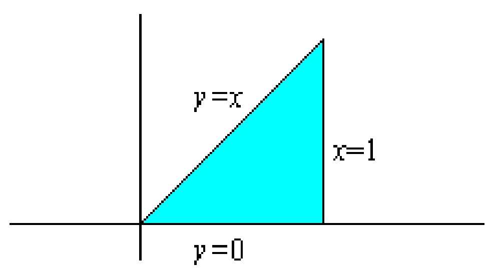


After combining this with the definition of $A(x)$, the result is the iterated integral

$$V = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\, dy \right] dx \qquad (1)$$

EXAMPLE 5 Find the volume of the solid under the graph of $f(x,y) = 2 - x^2 - y^2$ over the type I region

$x = 0 \quad y = 0$
$x = 1 \quad y = x$

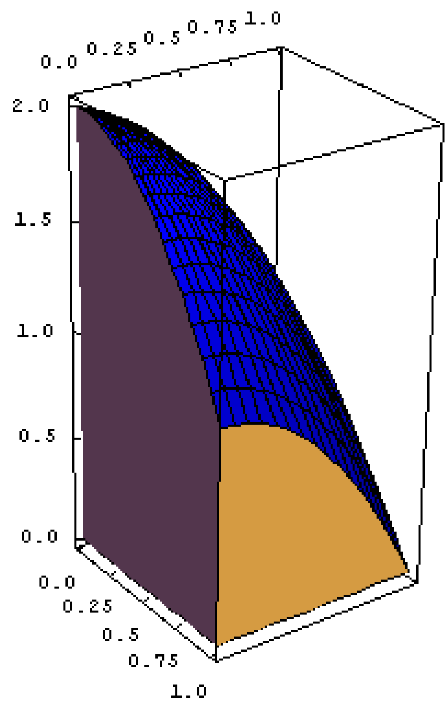


Solution: According to (1), the volume of the solid is

$$V = \int_0^1 \left[\int_0^x \left(2 - x^2 - y^2\right) dy \right] dx$$

We evaluate the resulting type I iterated integral by first evaluating the innermost integral:d

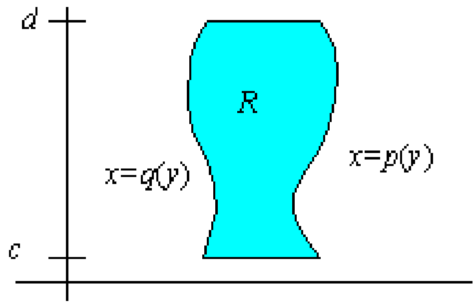
$$\begin{aligned} V &= \int_0^1 \left[2y - x^2 y - \frac{y^3}{3} \Big|_0^x \right] dx \\ &= \int_0^1 \left[2x - \frac{4}{3}x^3 \right] dx \\ &= \frac{2}{3} \end{aligned}$$



Check your Reading: Why is $2-x^2-y^2$ non-negative over the region bounded by $x=0$, $x=1$, $y=0$, $y=x$? Explain.

Volumes of Solids over Type II Regions

Similarly, if $p(y) \leq q(y)$ for y in $[c, d]$, then the region R in the xy-plane bounded by the curves $y = c$, $y = d$, $x = p(y)$, and $x = q(y)$,



is said to be a *type II region*. Correspondingly, if $f(x, y) \geq 0$ for all (x, y) in a type II region R, then the volume of the solid under $z = f(x, y)$ and over the region R is

$$V = \int_c^d \int_{p(y)}^{q(y)} f(x, y)\, dxdy \tag{2}$$

EXAMPLE 6 Find the volume of the solid under the graph of $f(x, y) = x^2 + y^2$ over the type II region

$$\begin{array}{ll} y = 0 & x = y^2 \\ y = 1 & x = y \end{array}$$

Solution: To do so, we use (2) to see that

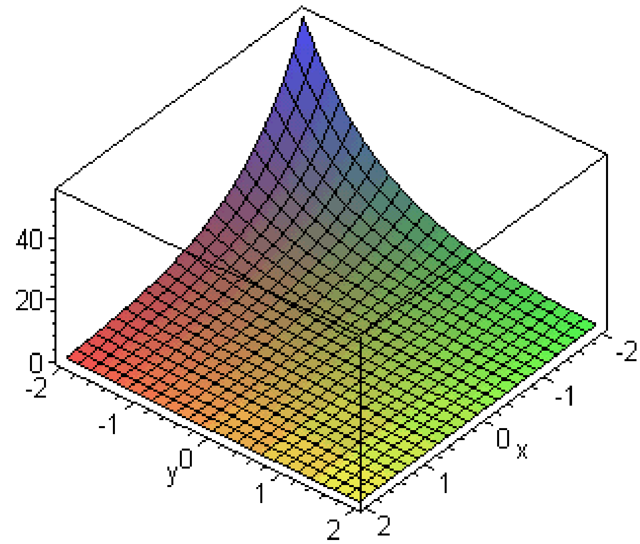
$$V = \int_0^1 \int_{y^2}^{y} \left(x^2 + y^2\right) dxdy$$

Evaluating the innermost integral leads to

$$\begin{aligned} V &= \int_0^1 \left[\frac{x^3}{3} + xy^2 \Big|_{y^2}^{y} \right] dy \\ &= \int_0^1 \left[\frac{4}{3}y^3 - \frac{1}{3}y^6 - y^4 \right] dy \\ &= \frac{3}{35} \end{aligned}$$

Finally, let us note that unbounded regions can lead to convergent improper integrals. Indeed, unbounded solids can have a finite volume.

EXAMPLE 7 Find the volume of the solid under the graph of $f(x,y) = e^{-x-y}$ over the first quadrant.



Solution: In the first quadrant, x is in $(0,\infty)$ and y is in $(0,\infty)$. Thus, (2) implies that

$$V = \int_0^\infty \int_0^\infty e^{-x-y} dy dx$$

The inner integral is evaluated as an improper integral

$$\begin{aligned} V &= \int_0^\infty \lim_{R\to\infty} \int_0^R e^{-x-y} dy dx \\ &= \int_0^\infty \lim_{R\to\infty} \left(e^{-x-0} - e^{-x-R}\right) dx \\ &= \int_0^\infty e^{-x} dx \end{aligned}$$

The resulting integral is also evaluated as an improper integral, leading to

$$V = \lim_{S\to\infty} \int_0^S e^{-x} dx = \lim_{S\to\infty} \left(e^0 - e^{-R}\right) = 1$$

Exercises

Identify each integral as either type I or type II and evaluate:

1.	$\int_0^1 \int_0^1 (x+y)\, dydx$	2.	$\int_0^2 \int_1^3 x^2 y\; dydx$
3.	$\int_0^2 \int_0^3 xy\; dxdy$	4.	$\int_0^1 \int_0^3 dydx$
5.	$\int_0^1 \int_0^x \left(x^2+y^2\right) dydx$	6.	$\int_0^\pi \int_0^{\sin(x)} dydx$
7.	$\int_0^\pi \int_0^\pi \cos(x)\, dydx$	8.	$\int_0^\pi \int_0^x \sin(y)\, dydx$
9.	$\int_0^{\pi/4} \int_0^{\sec(x)\tan(x)} dydx$	10.	$\int_0^{2\pi} \int_0^{\sin(x)} ydydx$
11.	$\int_0^\pi \int_0^x \sin(x)\, dydx$	12.	$\int_0^1 \int_0^y e^{x+y} dxdy$
13.	$\int_0^\pi \int_0^{\exp(x)} xdydx$	14.	$\int_0^1 \int_0^y \sin\left(y^2\right) dxdy$
15.	$\int_0^2 \int_0^y \ln\left(y^2+1\right) dxdy$	16.	$\int_0^3 \int_x^1 e^y dxdy$
17.	$\int_1^2 \int_0^{x^2} \frac{x}{x^2+y^2} dydx$	18.	$\int_1^2 \int_0^x \frac{1}{x^2+y^2} dydx$

Sketch the region R and determine its type. Then find the volume of the solid under $z = f(x,y)$ and over the given region.

19. $f(x,y) = x^2 + y^2$
 R: $y = 0,\ y = 1$
 $x = 0,\ x = 1$

20. $f(x,y) = 3$
 R: $x = 0,\ x = 2$
 $y = 0,\ y = 4$

21. $f(x,y) = 3x + 2y$
 R: $x = 0,\ x = 1$
 $y = 0,\ y = x^2$

22. $f(x,y) = 6x + y$
 R: $x = 2,\ x = 3$
 $y = 0,\ y = e^x$

23. $f(x,y) = xy$
 R: $y = 0,\ y = 1$
 $x = -y,\ x = y$

24. $f(x,y) = y^2$
 R: $y = 0,\ y = \pi/2$
 $x = 0,\ x = \sin(y)$

25. $f(x,y) = e^{x+y}$
 R: $y = 0,\ y = 1$
 $x = 0,\ x = 1 - y$

26. $f(x,y) = 9 - x^2 - y^2$
 R: $x = 1,\ x = 3$
 $y = x,\ y = x^2$

The following regions are unbounded. Sketch the region R and determine its type. Then find the volume of the solid under $z = f(x,y)$ and over the given region.

27. $f(x,y) = \frac{1}{x^2 y^2}$
 R: x in $(1,\infty)$, y in $(1,\infty)$

28. $f(x,y) = \frac{1}{x^2+y^2}$
 R: $x = 0,\ x = 2$

29. $f(x,y) = x^{-2} e^{-y}$
 R: x in $(1,\infty)$
 $y = 0,\ y = x^{-2}$

30. $f(x,y) = 1$
 R: x in $(0,\infty)$
 $y = x - e^{-x},\ y = x + e^{-x}$

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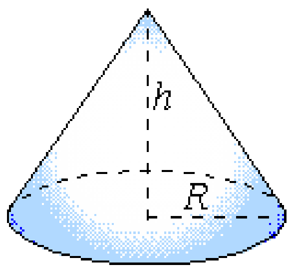
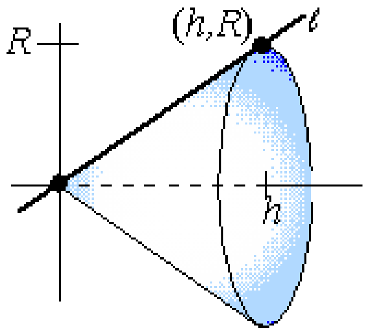
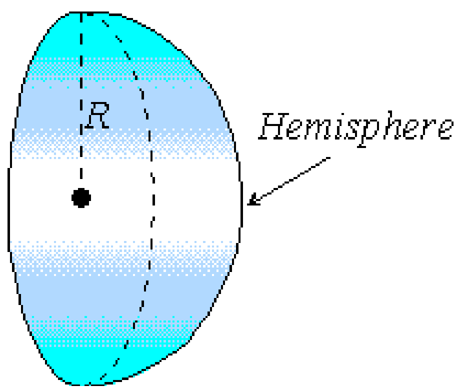
31. A regular cone with a height h and a base with radius R is positioned so that its axis is horizontal. Find the area $A(x)$ of a vertical cross-section of the cone perpendicular to the axis as a function of x in $[0, h]$.





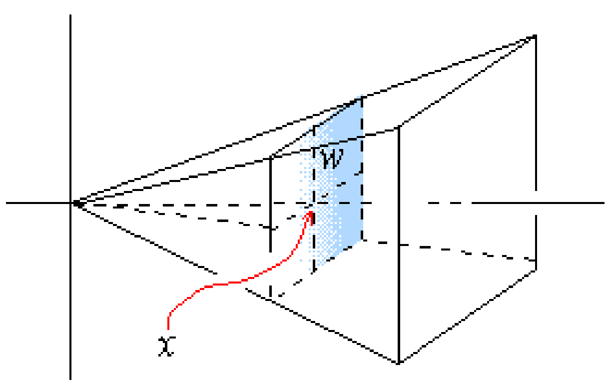
What is the volume of a regular cone with height h and a base with radius R?

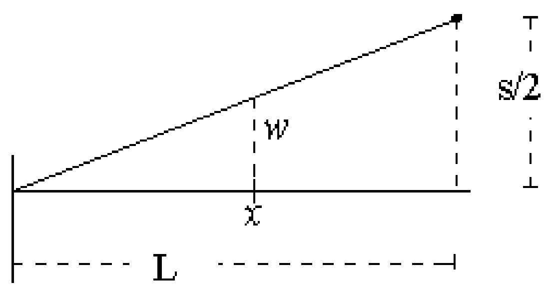
32. A hemisphere with radius R is positioned so that its axis is horizontal. Find the area $A(x)$ of a vertical cross-section of the cone perpendicular to the axis as a function of x in $[0, R]$.



What is the volume of a hemisphere with radius R?

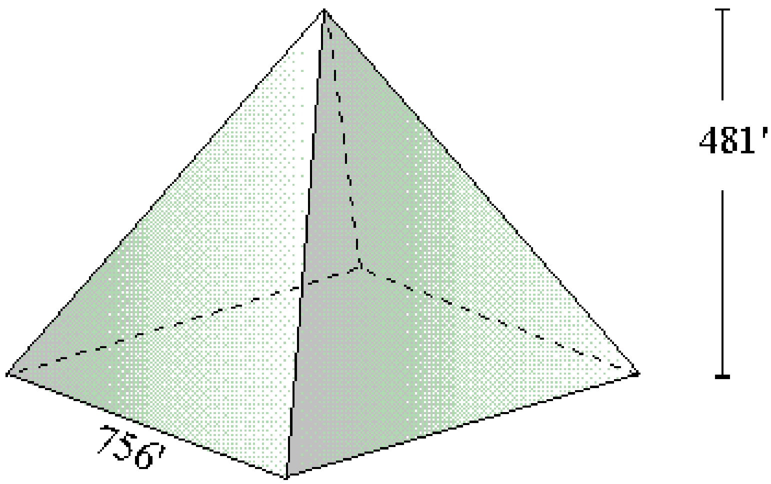
33. A regular pyramid has height h and a square base with each side a length s. It is positioned as shown in the figure below:





Find the area $A(x)$ of a cross-section at x. What is the volume of the pyramid?

34. The Great Pyramid is $481'$ tall and has a square base which is $756'$ wide on each side.



What is the volume of the Great Pyramid? (hint: see problem 33).

35. Explain why the area of a type I region can be written in the form

$$A = \int_a^b \int_{g(x)}^{h(x)} dydx$$

36. Explain why the area of a type II region can be written in the form

$$A = \int_c^d \int_{p(y)}^{q(y)} dxdy$$

37. Explain why if a, b, c, and d are all constant, then

$$\int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy$$

when both iterated integrals exist.

38. Show that if a, b, c, and d are constant, then

$$\int_a^b \int_c^d f(x)\, g(y)\, dydx = \left[\int_a^b f(x)\, dx\right] \left[\int_c^d g(y)\, dy\right]$$

39. Use properties of the integral to show that

$$\int_a^b \int_{p(x)}^{q(x)} [f(x, y) + g(x, y)]\, dy\, dx = \int_a^b \int_{p(x)}^{q(x)} f(x, y)\, dy\, dx + \int_a^b \int_{p(x)}^{q(x)} g(x, y)\, dy\, dx$$

40. Use properties of the integral to show that

$$\int_a^b \int_{p(x)}^{q(x)} [f(x,y) + g(x,y)]\, dy\, dx = \int_a^b \int_{p(x)}^{q(x)} f(x,y)\, dy\, dx + \int_a^b \int_{p(x)}^{q(x)} g(x,y)\, dy\, dx$$

41. Show that if f is differentiable on (a,b), then for all c in (a,b) we have

$$f(c)(b-a) + \int_a^b f(x)\, dx = \int_a^b \int_c^x f'(u)\, du dx$$

42. Show that if f is differentiable and if $f(0) = 0$, then

$$\int_a^b f(x)\, dx = \int_a^b \int_0^1 f'(ux)\, du dx$$