Practice Test

Chapter 3 Name ______________________________

Instructions. Show your work and/or explain your answers. (**Note:** Concepts from "DIFF GEOM" sections included only in the last problem).

1. Find the equation of the tangent plane to the level surface
$$x + z^2 = y + 1$$
at the point $(2, 2, 1)$.

   Solution: Since $x - y + z^2 = 1$, we let $U(x, y, z) = x - y + z^2$. Thus,
$$\nabla U = \langle 1, -1, 2z \rangle$$
The normal to the surface is $\mathbf{n} = \langle 1, -1, 2 \rangle$ and the equation is
$$1(x - 2) - 1(y - 2) + 2(z - 1) = 0$$
which results in $z = -\frac{1}{2}x + \frac{1}{2}y + 1$.

2. Find the level surface representation of the parametric surface
$$\mathbf{r}(u, v) = \left\langle v\sin(u), v^2, v\cos(u) \right\rangle$$
Solution: Clearly, $x = v\sin(u)$, $y = v^2$, and $z = v\cos(u)$. We can eliminate u by noticing that
$$x^2 + z^2 = v^2\sin^2(u) + v^2\cos^2(u) = v^2$$
Since $y = v^2$, we eliminate v by noticing that $x^2 + z^2 = y$.

3. Find the level surface representation of the parametric surface
$$\mathbf{r}(u, v) = \left\langle e^u\cosh(v), e^u\sinh(v), e^{-u} \right\rangle$$
Solution: Clearly, $x = e^u\cosh(v)$, $y = e^u\sinh(v)$, and $z = e^{-u}$. To eliminate v we notice that
$$x^2 - y^2 = e^{2u}\cosh^2(v) - e^{2u}\sinh^2(v) = e^{2u}\left[\cosh^2(v) - \sinh^2(v)\right] = e^{2u}$$
To eliminate u, we notice that $z^2 = e^{-2u}$, so that
$$\left(x^2 - y^2\right)z^2 = e^{2u}e^{-2u} = 1$$
so that the surface is given by $x^2z^2 - y^2z^2 = 1$.

4. Find the parametric equation of the tangent plane to the parametric surface
$$\mathbf{r}(u, v) = \left\langle v\sin(u), v^2, v\cos(u) \right\rangle$$
at $(u, v) = \left(\frac{\pi}{4}, 1\right)$.

   Solution: $\mathbf{r}_u = \langle v\cos(u), 0, -v\sin(u) \rangle$ and $\mathbf{r}_v = \langle \sin(u), 2v, \cos(u) \rangle$. Thus, we have
$$\mathbf{r}_u(\pi/4, 1) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle \quad \textit{and} \quad \mathbf{r}_v(\pi/4, 1) = \left\langle \frac{1}{\sqrt{2}}, 2, \frac{1}{\sqrt{2}} \right\rangle$$
Since $\mathbf{r}(\pi/4, 1) = \left\langle 1/\sqrt{2}, 1, 1/\sqrt{2} \right\rangle$, the parameterization of the tangent plane is
$$\begin{aligned} \mathbf{L}(du, dv) &= \mathbf{r}(\pi/4, 1) + \mathbf{r}_u(\pi/4, 1)\,du + \mathbf{r}_v(\pi/4, 1)\,dv \\ &= \left\langle \frac{1}{\sqrt{2}}, 1, \frac{-1}{\sqrt{2}} \right\rangle + \left\langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle du + \left\langle \frac{1}{\sqrt{2}}, 2, \frac{1}{\sqrt{2}} \right\rangle dv \\ &= \left\langle \frac{1}{\sqrt{2}} + \frac{du}{\sqrt{2}} + \frac{dv}{\sqrt{2}}, 1 + 2dv, \frac{-1}{\sqrt{2}} - \frac{du}{\sqrt{2}} + \frac{dv}{\sqrt{2}} \right\rangle \end{aligned}$$

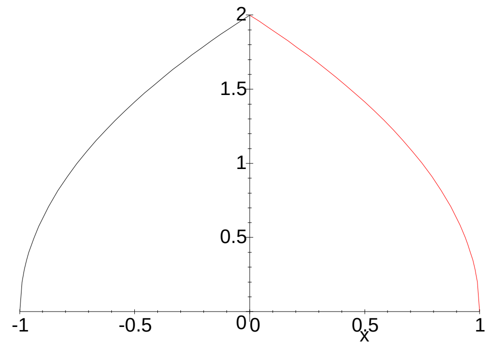
5. Find the image of the unit square under the coordinate transformation

$$T(u,v) = \langle u^2 - v^2, 2uv \rangle$$

Solution: Since $x = u^2 - v^2$ and $y = 2uv$, we have

$$\begin{aligned} v &= 0: \quad x = u^2, y = 0 \Longrightarrow x \geq 0 \\ u &= 1: \quad x = 1 - v^2, y = 2v \Longrightarrow x = 1 - \frac{y^2}{4} \\ v &= 1: \quad x = u^2 - 1, y = 2u \Longrightarrow x = \frac{y^2}{4} - 1 \\ u &= 0: \quad x = -v^2, y = 0 \Longrightarrow x \leq 0 \end{aligned}$$

Putting them all together yields the following region:



6. Find the matrix of rotation through an angle of $\theta = 45°$. Then use this to rotate the line $v = u + 1$ through an angle of $45°$.

Solution: The rotation matrix is

$$R_{\pi/4} = \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Thus, rotation of the vector $\langle u, v \rangle$ results in

$$T(u,v) = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2}u - \frac{1}{2}\sqrt{2}v \\ \frac{1}{2}\sqrt{2}u + \frac{1}{2}\sqrt{2}v \end{bmatrix}$$

That is, $x = \frac{1}{2}\sqrt{2}u - \frac{1}{2}\sqrt{2}v$ and $y = \frac{1}{2}\sqrt{2}u + \frac{1}{2}\sqrt{2}v$. Since $v = u + 1$, we get

$$\begin{aligned} x &= \frac{1}{2}\sqrt{2}u - \frac{1}{2}\sqrt{2}(u+1) = -\frac{\sqrt{2}}{2} \\ y &= y = \frac{1}{2}\sqrt{2}u + \frac{1}{2}\sqrt{2}(u+1) = \sqrt{2}u + \frac{1}{2}\sqrt{2} \end{aligned}$$

Thus, $v = u + 1$ is mapped to the line $x = \frac{-\sqrt{2}}{2}$.

7. Convert the following into polar coordinates and solve for r :

$$y = 3x + 1$$

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$r\sin(\theta) = 3r\cos(\theta) + 1, \qquad r = \frac{1}{\sin\theta - 3\cos\theta}$$

8. Convert the following into polar coordinates and solve for r :

$$x^2 + y^2 = x + y$$

Solution: $r^2 = r\cos(\theta) + r\sin(\theta)$ implies that

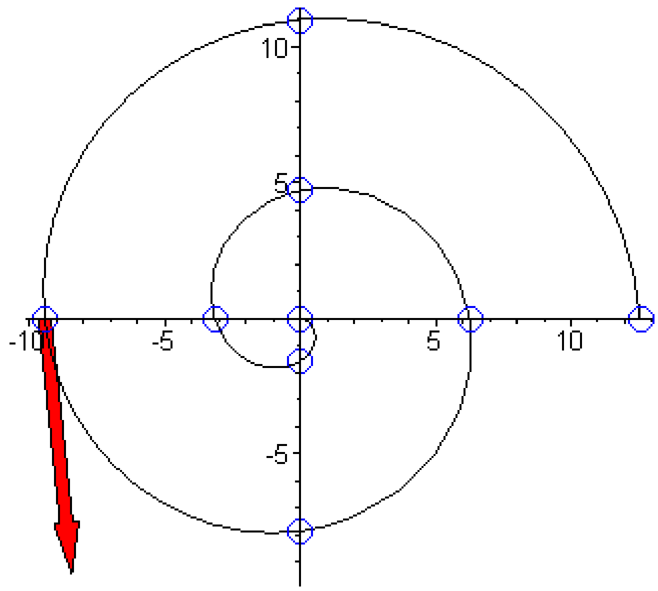
$$r = \cos(\theta) + \sin(\theta)$$

9. Sketch the graph of $r = 4\pi - \theta$ in polar coordinates when θ is in $[0, 4\pi]$. Then find and sketch the tangent vector to the curve when $\theta = \pi$

Solution: To sketch the curve, we use a table of values of r versus θ :

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
r	$4\pi \approx 12.57$	$\frac{7\pi}{2} \approx 11.0$	$3\pi \approx 9.42$	$\frac{5\pi}{2} \approx 7.85$	$2\pi \approx 6.28$	$\frac{3\pi}{2} \approx 4.71$	$\pi \approx 3.14$	$\frac{\pi}{2} \approx 1.57$	0

We plot the points in polar coordinates to obtain the curve.



The Jacobian in polar coordinates yields the tangent vector:

$$\mathbf{v} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} r'(\theta) \\ 1 \end{bmatrix}$$

Since $r(\pi) = 3\pi$ and $r'(\theta) = -1$, we have

$$\mathbf{v} = \begin{bmatrix} \cos(\pi) & -3\pi\sin(\pi) \\ \sin(\pi) & 3\pi\cos(\pi) \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3\pi \end{bmatrix}$$

That is, the tangent vector is $\mathbf{v} = \langle 1, -3\pi \rangle$, which is shown in the plot above.

10. Find the Jacobian determinant and area differential of the coordinate transformation

$$T(u,v) = \langle u - v, u^2 + v^2 \rangle$$

Solution: Since $x = u - v$ and $y = u^2 + v^2$, the jacobian is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= 1 \cdot 2v - (-1) \cdot 2u \\ &= 2u + 2v \end{aligned}$$

Thus, the area differential is $dA = |2u + 2v|\, dudv$.

11. What is the unit surface normal for the surface $x^2+y^2+z^2=2x$? What is the unit surface normal for the surface in *cylindrical* coordinates?

Solution: Writing $x^2+y^2+z^2-2x=0$ leads us to define $U=x^2+y^2+z^2-2x$. Thus, $\nabla U = \langle 2x-2, 2y, 2z\rangle$ and

$$\begin{aligned}\|\nabla U\| &= \sqrt{(2x-2)^2+4y^2+4z^2}\\ &= \sqrt{4x^2-8x+4+4y^2+4z^2}\\ &= \sqrt{4\left(x^2-2x+y^2+z^2\right)+4}\\ &= \sqrt{4(0)+4}\\ &= 2\end{aligned}$$

Thus, the unit surface normal is

$$\mathbf{n}=\frac{\nabla U}{\|\nabla U\|}=\langle x-1,y,z\rangle \tag{1}$$

In cylindrical coordinates, $U=r^2+z^2-2r\cos(\theta)$ and the gradient is given by

$$\nabla U = U_r\mathbf{e}_r+\frac{1}{r}U_\theta\mathbf{e}_\theta+U_z\mathbf{k}$$

where $\mathbf{e}_r=\langle\cos(\theta),\sin(\theta),0\rangle$ and $\mathbf{e}_\theta=\langle-\sin(\theta),\cos(\theta),0\rangle$. Thus, in cylindrical coordinates, we have

$$\begin{aligned}\nabla U &= (2r-2\cos(\theta))\langle\cos(\theta),\sin(\theta),0\rangle+\frac{1}{r}(2r\sin(\theta))\langle-\sin(\theta),\cos(\theta),0\rangle+2z\langle 0,0,1\rangle\\ &= \left\langle 2r\cos(\theta)-2\cos^2(\theta),2r\sin(\theta)-2\cos(\theta)\sin(\theta),0\right\rangle+\left\langle -2\sin^2(\theta),2\sin(\theta)\cos(\theta),0\right\rangle+\langle 0,0,2z\rangle\\ &= \left\langle 2r\cos(\theta)-2\left(\cos^2(\theta)+\sin^2(\theta)\right),2r\sin(\theta)-2\cos(\theta)\sin(\theta)+2\cos(\theta)\sin(\theta),2z\right\rangle\\ &= \langle 2r\cos(\theta)-2,2r\sin(\theta),2z\rangle\end{aligned}$$

That is, $\nabla U=\langle 2r\cos(\theta)-2,2r\sin(\theta),2z\rangle$, and as shown above, $\|\nabla U\|=2$, so that

$$\mathbf{n}=\frac{\nabla U}{\|\nabla U\|}=\langle r\cos(\theta)-1,r\sin(\theta),z\rangle$$

Notice that you could have obtained the same result by simply applying cylindrical coordinates to (1).

12. Find the pullback of the surface $x^2+y^2+z^2=2x$ into spherical coordinates, and then use the result to construct a parameterization of the surface.

Solution: In spherical coordinates, $\rho^2=x^2+y^2+z^2$, $r=\rho\sin(\phi)$ and $x=r\cos(\theta)=\rho\sin(\phi)\cos(\theta)$. Thus, the surface becomes

$$\rho^2=\rho\sin(\phi)\cos(\theta)\quad or\quad \rho=\sin(\phi)\cos(\theta)\quad for\ \rho\neq 0$$

Since also $y=\rho\sin(\phi)\sin(\theta)$ and $z=\rho\cos(\phi)$, the parameterization of the surface is

$$\begin{aligned}\mathbf{r}(\phi,\theta) &= \langle\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta),\rho\cos(\phi)\rangle\\ &= \left\langle\sin^2(\phi)\cos^2(\theta),\sin^2(\phi)\cos(\theta)\sin(\theta),\sin(\phi)\cos(\phi)\cos(\theta)\right\rangle\end{aligned}$$

13. Use the fundamental form of the plane in polar coordinates to find the length of the polar curve $r=e^{-\theta/4}$, θ in $[0,2\pi]$.

Solution: The fundamental form of the plane in polar coordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2$$

which leads to an arclength formula of

$$L = \int_a^b \frac{ds}{d\theta}\, d\theta = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Substituting $r = e^{-\theta/4}$ and $r' = -e^{-\theta/4}/4$ leads to

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{e^{-\theta/2} + \frac{1}{16} e^{-\theta/2}}\, d\theta \\ &= \int_0^{2\pi} e^{-\theta/4} \sqrt{1 + \frac{1}{16}}\, d\theta \\ &= \sqrt{\frac{17}{16}}\, \frac{e^{-\theta/4}}{-1/4}\Bigg|_0^{2\pi} \\ &= -e^{-\frac{1}{2}\pi}\sqrt{17} + \sqrt{17} \end{aligned}$$

14. Find the fundamental form of the surface $\mathbf{r}(u, v) = \langle v \sin(u), v \cos(u), v \rangle$. Then use it to compute the arclength of

$$v = \sin\left(\frac{u}{\sqrt{2}}\right), \quad u \ in \ \left[0, \frac{\pi}{4}\right]$$

Solution: $\mathbf{r}_u = \langle v \cos(u), -v \sin(u), 0 \rangle$ and $\mathbf{r}_v = \langle \sin(u), \cos(u), 1 \rangle$. Thus,

$$\begin{aligned} g_{11} &= \mathbf{r}_u \cdot \mathbf{r}_u = v^2 \cos^2(u) + v^2 \sin^2(u) = v^2 \\ g_{12} &= \mathbf{r}_u \cdot \mathbf{r}_v = v \cos(u) \sin(u) - v \sin(u) \cos(u) = 0 \\ g_{22} &= \mathbf{r}_v \cdot \mathbf{r}_v = \sin^2(u) + \cos^2(u) + 1 = 2 \end{aligned}$$

Thus, the fundamental form is

$$(ds)^2 = v^2 (du)^2 + 2 (dv)^2$$

and correspondingly, the arclength integral is

$$L = \int_0^{\pi/4} \sqrt{v^2 + 2\left(\frac{dv}{du}\right)^2}\, du$$

Since $dv/du = \cos\left(u/\sqrt{2}\right)/\sqrt{2}$, the arclength is

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{\sin^2\left(\frac{u}{\sqrt{2}}\right) + \frac{2}{\left(\sqrt{2}\right)^2} \cos^2\left(\frac{u}{\sqrt{2}}\right)}\, du \\ &= \int_0^{\pi/4} \sqrt{\sin^2\left(\frac{u}{\sqrt{2}}\right) + \cos^2\left(\frac{u}{\sqrt{2}}\right)}\, du \\ &= \int_0^{\pi/4} du = \frac{\pi}{4} \end{aligned}$$

15. DIFF GEOM: For the right circular cone, $\mathbf{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), r \rangle$, do the following:

    (a) Show that curves of the form $\theta = k$ for k constant are geodesics on the cone. What are these curves?

    (b) Find the fundamental form of the cone and calculate the shortest distance between the points with coordinates $(r, \theta) = (1, \pi)$ and $(r, \theta) = (3, \pi)$.

(c) What are the principal curvatures of the surface?

(d) What are the mean and Gaussian curvatures of the surface? Do you obtain the same Gaussian curvature if you use the theorem Egregium?

Solution: (a) Curves of the form $\theta = k$ are parameterized by

$$\boldsymbol{\rho}(t) = \langle r(t)\cos(k), r(t)\sin(k), r(t)\rangle$$

for some unknown function $r(t)$, and the acceleration is $\boldsymbol{\rho}''(t) = \langle r''\cos(k), r''\sin(k), r''\rangle$. However, $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), 1\rangle$ and $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$. Clearly, we have

$$\boldsymbol{\rho}'' \cdot \mathbf{r}_r = r''\cos^2(k) + r''\sin^2(k) + r'' = r'' \qquad \textit{and} \qquad \boldsymbol{\rho}'' \cdot \mathbf{r}_\theta = 0$$

If $r'' = 0$, then $r(t) = pt + c$ for constants p and c. Thus, $\theta = k$ constant and $r = mt + b$ for m, b constant parameterizes a geodesic, which implies that $\theta = k$ for k constant is a geodesic. These are straight lines on the cone that pass through the origin.

(b) Since $\mathbf{r}_r = \langle \cos(\theta), \sin(\theta), 1\rangle$ and $\mathbf{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$, we have $g_{11} = \mathbf{r}_r \cdot \mathbf{r}_r = 2$ and $g_{22} = \mathbf{r}_\theta \cdot \mathbf{r}_\theta = r^2$ with $g_{12} = 0$ (the parameterization is orthogonal). Thus, the fundamental form is

$$ds^2 = 2dr^2 + r^2 d\theta^2$$

However, on $\theta = k$, the differential $d\theta = 0$, so that we have

$$ds = \sqrt{2}dr$$

Since r ranges from 1 to 3, this implies that

$$L = \int_1^3 \sqrt{2}dr = 2\sqrt{2}$$

(c) Since $\mathbf{r}_r \times \mathbf{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle$, the surface normal is

$$\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|} = \frac{1}{\sqrt{2}}\langle -\cos(\theta), -\sin(\theta), 1\rangle$$

Since $\mathbf{r}_{rr} = \mathbf{0}$, $\mathbf{r}_{r\theta} = \langle -\sin(\theta), \cos(\theta), 0\rangle$ and $\mathbf{r}_{\theta\theta} = \langle -r\cos(\theta), -r\sin(\theta), 0\rangle$, the normal curvature of the cone is given by

$$\begin{aligned}
\kappa_n(\xi) &= \frac{\mathbf{r}_{rr}\cdot\mathbf{n}}{\|\mathbf{r}_r\|^2}\cos^2(\xi) + \frac{\mathbf{r}_{r\theta}\cdot\mathbf{n}}{\|\mathbf{r}_r\|\,\|\mathbf{r}_\theta\|}\sin(2\xi) + \frac{\mathbf{r}_{\theta\theta}\cdot\mathbf{n}}{\|\mathbf{r}_\theta\|^2}\sin^2(\xi) \\
&= \frac{\langle -\sin(\theta), \cos(\theta), 0\rangle \cdot \langle -\cos(\theta), -\sin(\theta), 1\rangle}{r\sqrt{2}}\sin(2\xi) + \frac{\langle -r\cos(\theta), -r\sin(\theta), 0\rangle \cdot \langle -\cos(\theta), -\sin(\theta), 1\rangle}{r^2\sqrt{2}}\sin \\
&= \frac{r}{r^2\sqrt{2}}\sin^2(\xi) \\
&= \frac{1}{r\sqrt{2}}\sin^2(\xi)
\end{aligned}$$

Thus, the principal curvatures occur when $\xi = 0$ and when $\xi = \pi/2$, and are correspondingly

$$\kappa_1 = 0 \qquad \textit{and} \qquad \kappa_2 = \frac{1}{r\sqrt{2}}$$

(d) The mean curvature is

$$H = \kappa_1 + \kappa_2 = \frac{1}{r\sqrt{2}}$$

and the Gaussian curvature is 0 since $\kappa_1 = 0$. Using the theorem Egregium, we would obtain

$$K = \frac{-1}{2\sqrt{g}}\left[\frac{\partial}{\partial\theta}\left(\frac{g_{11,\theta}}{\sqrt{g}}\right) + \frac{\partial}{\partial r}\left(\frac{g_{22,r}}{\sqrt{g}}\right)\right]$$

where $g_{11} = 2$, $g_{22} = r^2$, and $g = g_{11}g_{22} = 2r^2$. Thus,

$$\begin{aligned} K &= \frac{-1}{2r\sqrt{2}}\left[\frac{\partial}{\partial\theta}\left(\frac{0}{\sqrt{g}}\right) + \frac{\partial}{\partial r}\left(\frac{2r}{r\sqrt{2}}\right)\right] \\ &= \frac{-1}{2r\sqrt{2}}\left[\frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial r}\left(\frac{2}{\sqrt{2}}\right)\right] \\ &= 0 \end{aligned}$$

Thus, the cone is a surface that can be "made with a piece of paper" without stretching or tearing the paper.