Surface Normals and Tangent Planes

Normal and Tangent Planes to Level Surfaces

Because the equation of a plane requires a point and a normal vector to the plane, finding the equation of a tangent plane to a surface at a given point requires the calculation of a *surface normal vector*. In this section, we explore the concept of a normal vector to a surface and its use in finding equations of tangent planes.

To begin with, a level surface $U(x,y,z) = k$ is said to be *smooth* if the *gradient* $\nabla U = \langle U_x, U_y, U_z \rangle$ is continuous and non-zero at any point on the surface. Equivalently, we often write

$$\nabla U = U_x \mathbf{e}_x + U_y \mathbf{e}_y + U_z \mathbf{e}_z$$

where $\mathbf{e}_x = \langle 1,0,0 \rangle$, $\mathbf{e}_y = \langle 0,1,0 \rangle$, and $\mathbf{e}_z = \langle 0,0,1 \rangle$.

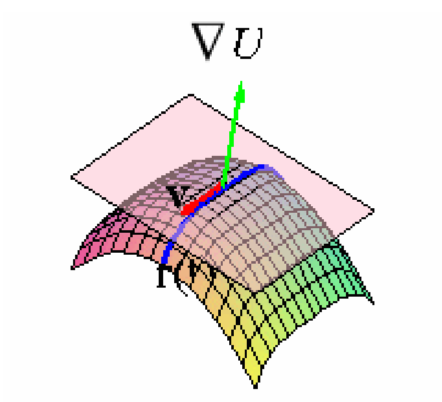
Suppose now that $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ lies on a smooth surface $U(x,y,z) = k$. Applying the derivative with respect to t to both sides of the equation of the level surface yields

$$\frac{dU}{dt} = \frac{d}{dt}k$$

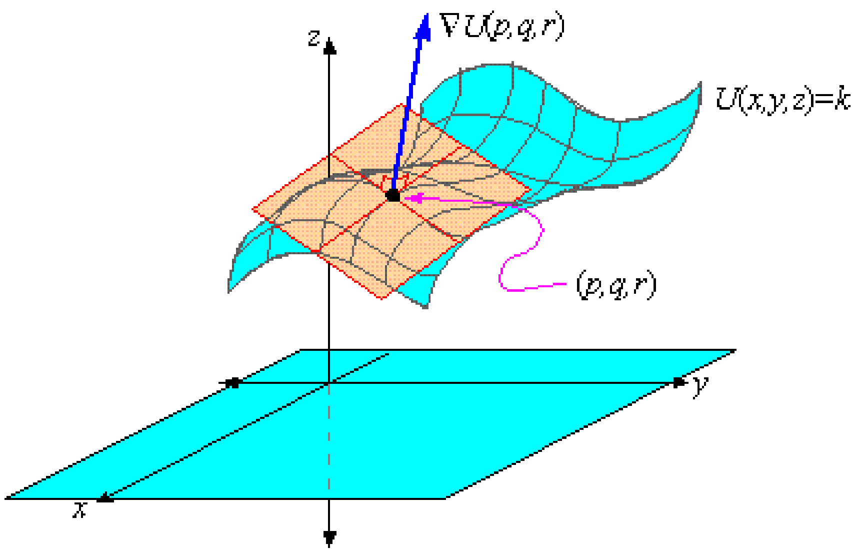
Since k is a constant, the chain rule implies that

$$\nabla U \cdot \mathbf{v} = 0$$

where $\mathbf{v} = \langle x'(t), y'(t), z'(t) \rangle$. However, $\mathbf{v}$ is tangent to the surface because it is tangent to a curve on the surface, which implies that ∇U is orthogonal to each tangent vector $\mathbf{v}$ at a given point on the surface.



That is, $\nabla U(p,q,r)$ at a given point (p,q,r) is *normal* to the tangent plane to the surface $U(x,y,z)=k$ at the point (p,q,r).



We thus say that the gradient ∇U is *normal* to the surface $U(x,y,z)=k$ at each point on the surface.

EXAMPLE 1 Find the equation of the tangent plane to the hyperboloid in 2 sheets

$$x^2-y^2-z^2=4$$

at the point $(3,2,1)$.

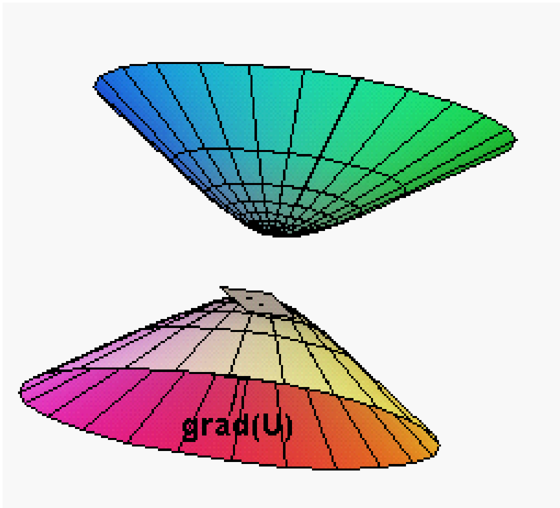
Solution: To begin with, we identify $U(x,y,z)=x^2-y^2-z^2$, so that its gradient is

$$\nabla U=\langle 2x,-2y,-2z\rangle$$

At the point $(2,3,1)$, the vector $\mathbf{n}=\nabla U(3,2,1)=\langle 6,-4,-2\rangle$ is normal to the surface, so that the equation of the tangent plane is

$$6(x-3)-4(y-2)-2(z-1)=0$$

which simplifies to the equation $z = 3x - 2y - 4$.



EXAMPLE 2 Find the equation of the tangent plane to the *right circular cone*

$$x^2 + y^2 = z^2$$

at the point $(0.6, 0.8, 1)$.

Solution: Since the equation of the surface can be written $x^2 + y^2 - z^2 = 0$, we let

$$U(x, y, z) = x^2 + y^2 - z^2$$

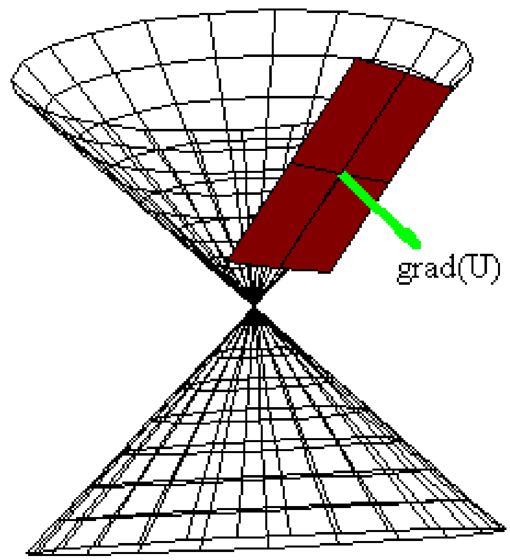
As a result, the gradient of U is

$$\nabla U = \langle 2x, 2y, -2z \rangle$$

At the point $(0.6, 0.8, 1)$, the vector $\nabla U(0.6, 0.8, 1) = \langle 1.2, 1.6, 2 \rangle$ is normal to the surface, so the equation of the tangent plane is

$$1.2(x - 0.6) + 1.6(y - 0.8) - 2(z - 1) = 0$$

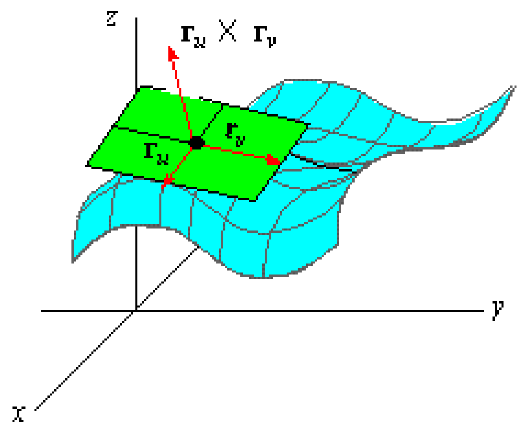
Solving for z yields $z = 0.6x + 0.8y$, which is shown in the figure below:



Check your Reading: What *degenerate conic section* is formed by the intersection of the cone with the plane in example 2?

Normal and Tangent Planes to Parametric Surfaces

If $\mathbf{r}(u,v)$ is a regular parametrization of a surface, then the vector $\mathbf{r}_u \times \mathbf{r}_v$ is perpendicular to both $\mathbf{r}_u$ and $\mathbf{r}_v$. Thus, $\mathbf{r}_u \times \mathbf{r}_v$ must also be perpendicular to the tangent plane spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$.



We say that the cross product $\mathbf{r}_u \times \mathbf{r}_v$ is *normal to the surface.* Similar to the first section, the vector $\mathbf{r}_u \times \mathbf{r}_v$ can be used as the normal vector in determining the equation of the tangent plane at a point of the form $(x_1, y_1, z_1) = \mathbf{r}(p, q)$.

EXAMPLE 3 Find the equation of the tangent plane to the *torus*

$$\mathbf{r} = \langle (2 + \sin(v)) \cos(u), (2 + \sin(v)) \sin(u), \cos(v) \rangle$$

at the point $\mathbf{r}(0, 0)$.

Solution: The vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are given by

$$\begin{aligned} \mathbf{r}_u &= \langle -(2 + \sin(v)) \sin(u), (2 + \sin(v)) \cos(u), 0 \rangle \\ \mathbf{r}_v &= \langle \cos(v) \cos(u), \cos(v) \sin(u), -\sin(v) \rangle \end{aligned}$$

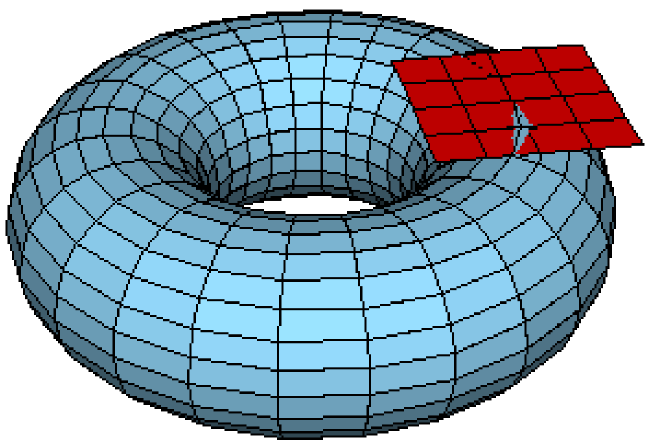
so that $\mathbf{r}_u(0, 0) = \langle 0, 2, 0 \rangle = 2\mathbf{j}$ and $\mathbf{r}_v(0, 0) = \langle 1, 0, 0 \rangle = \mathbf{i}$. Thus, the normal to the plane is

$$\mathbf{n} = \mathbf{r}_u(0, 0) \times \mathbf{r}_v(0, 0) = 2\mathbf{j} \times \mathbf{i} = -2\mathbf{k}$$

That is, $\mathbf{n} = \langle 0, 0, -2 \rangle$. Since $\mathbf{r}(0, 0) = (2, 0, 1)$, the equation of the tangent plane at that point is

$$0(x - 2) + 0(y - 0) - 2(z - 1) = 0$$

which reduces to $z = 1$.



If $\mathbf{r}(u,v)$ is the parameterization of a surface, then the *surface unit normal* is defined

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$$

The vector $\mathbf{n}$ is also normal to the surface.

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Moreover, $\mathbf{n}$ is often considered to be a function $\mathbf{n}(u,v)$ which assigns a normal unit vector to each point on the surface.

EXAMPLE 4 Find the surface unit normal and the equation of the tangent plane to the cylinder

$$\mathbf{r}(u,v) = \langle 3\cos(u), 3\sin(u), v\rangle$$

at $\mathbf{r}(\pi,2) = (-3,0,2)$.

Solution: Since $\mathbf{r}_u = \langle -3\sin(u), 3\cos(u), 0\rangle = -3\sin(u)\,\mathbf{i} + 3\cos(u)\,\mathbf{j}$ and since $\mathbf{r}_v = \langle 0,0,1\rangle = \mathbf{k}$, their cross product is

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= (-3\sin(u)\,\mathbf{i} + 3\cos(u)\,\mathbf{j}) \times \mathbf{k} \\ &= -3\sin(u)\,\mathbf{i}\times\mathbf{k} + 3\cos(u)\,\mathbf{j}\times\mathbf{k} \\ &= 3\sin(u)\,\mathbf{j} + 3\cos(u)\,\mathbf{i} \end{aligned}$$

It is easy to show that $||\mathbf{r}_u \times \mathbf{r}_v|| = 3$, so that the *unit surface normal* is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||} = \cos(u)\,\mathbf{i} + \sin(u)\,\mathbf{j}$$

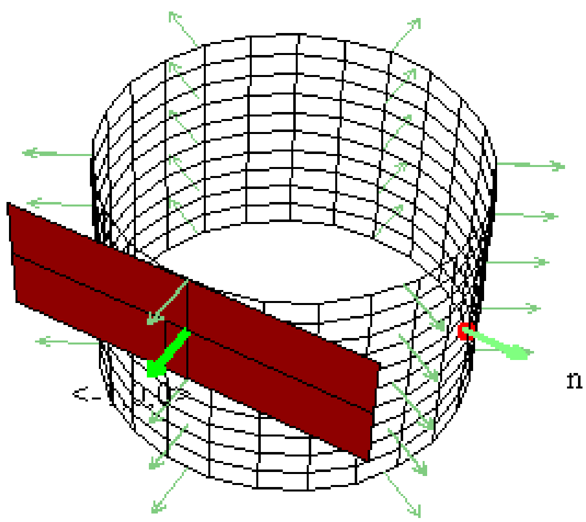
The tangent plane is the plane through $(-3, 0, 2)$ with a normal of

$$\mathbf{n}(\pi, 2) = \cos(\pi)\,\mathbf{i} + \sin(\pi)\,\mathbf{j} = \langle -1, 0, 0 \rangle$$

Thus, the equation of the tangent plane is

$$-1(x - -1) + 0(y - 0) + 0(z - 0) = 0$$

which reduces to $x = -1$.



Check Your Reading: Does the tangent plane in example 4 intersect the cylinder at only one point?

Tangents and Normals to Graphs of Functions

If $f(x, y)$ is differentiable at (p, q), then linearization of $f(x, y)$ at (p, q) leads to a tangent plane to $z = f(x, y)$ at $(p, q, f(p, q))$ of

$$z = f(p, q) + f_x(p, q)(x - p) + f_y(p, q)(y - q)$$

This formula was developed using the total derivative.

However, we can also find the tangent plane to $z = f(x, y)$ at $(p, q, f(p, q))$ by (a) considering the level surface $z - f(x, y) = 0$ or (b) considering the parameterization

$$\mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle$$

at the point $\mathbf{r}(p, q)$. The result is the same for all 3 methods.

EXAMPLE 5 Find the equation of the tangent plane to $f(x, y) = x^2 - y^2$ at $(1, 2)$ by (a) linearization of $f(x, y)$ at $(1, 2)$ (b) considering the surface $f(x, y) - z = 0$ at $(1, 2, f(1, 2))$ and (c) finding the tangent plane to a parameterization of the form

$$\mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle$$

at $\mathbf{r}(1, 2)$. The answer should be the same in all 3 cases.

Solution: (a) Since $f_x = 2x$ and $f_y = -2y$, we have $f_x(1, 2) = 2$, and $f_y(1, 2) = -4$. Thus, $f(1, 2) = -3$ and correspondingly, the equation of the tangent plane is

$$\begin{aligned} z &= f(p, q) + f_x(p, q)(x - p) + f_y(p, q)(y - q) \\ &= -3 + 2(x - 1) - 4(y - 2) \end{aligned}$$

which simplifies to $z = 2x - 4y + 3$. (b) If $U(x, y, z) = z - \left(x^2 - y^2\right)$, then

$$\nabla U = \langle -2x, 2y, 1 \rangle$$

Since $f(1, 2) = -3$, the point of tangency is $(1, 2, -3)$ and

$$\nabla U(1, 2, -3) = \langle -2, 4, 1 \rangle$$

Using $\nabla U(1, 2 - 3)$ as the normal, we obtain

$$-2(x - 1) + 4(y - 2) + 1(z - -3) = 0$$

which upon solving for z yields $z = 2x - 4y + 3$. (c) If we let

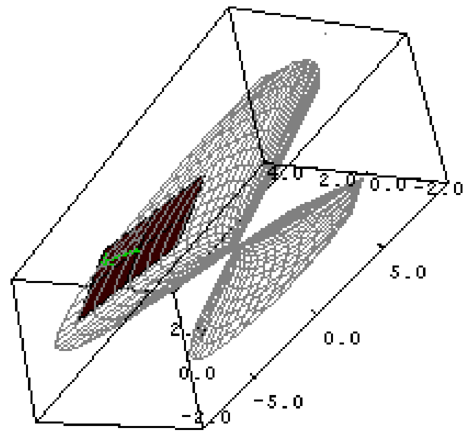
$$\mathbf{r}(u, v) = \left\langle u, v, u^2 - v^2 \right\rangle$$

then $\mathbf{r}_u = \langle 1, 0, 2u \rangle$ and $\mathbf{r}_v = \langle 0, 1, -2v \rangle$. Thus,

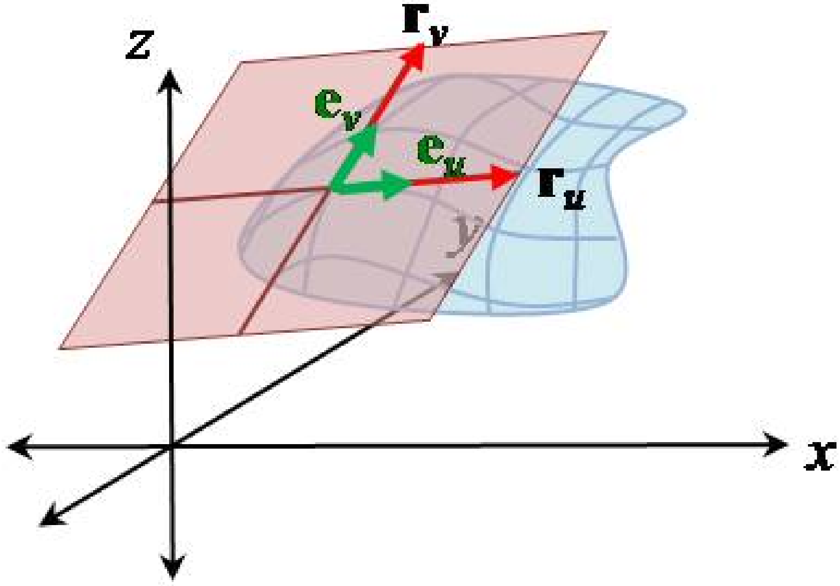
$$\mathbf{r}_u(1, 2) = \langle 1, 0, 2 \rangle \quad \text{and} \quad \mathbf{r}_v(1, 2) = \langle 0, 1, -4 \rangle$$

and $\mathbf{r}_u(1, 2) \times \mathbf{r}_v(1, 2) = \langle -2, 4, 1 \rangle$, which is the same as $\nabla U(1, 2, -3)$. Since $\mathbf{r}(1, 2) = \langle 1, 2, -3 \rangle$, the equation of the tangent plane is again

$$z = 2x - 4y + 3$$



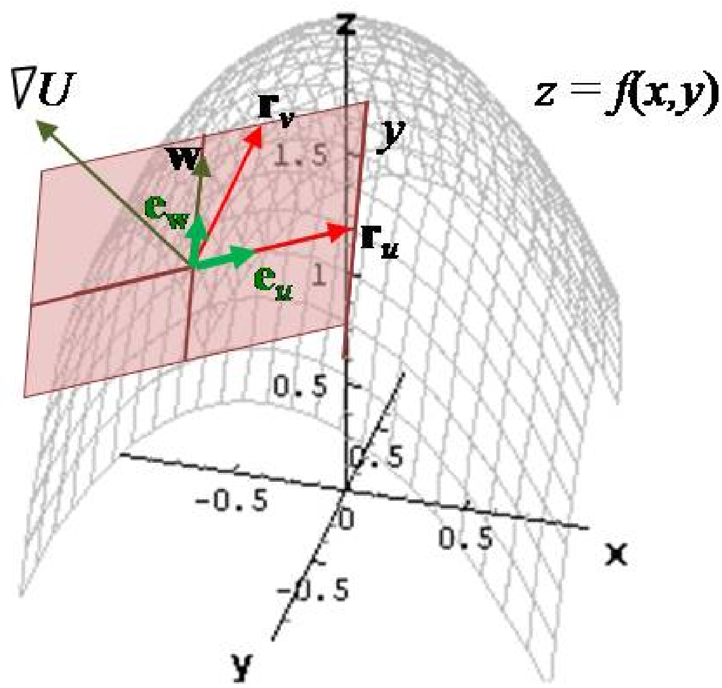
Of course, in applications only one approach is necessary. But there are times when one method is preferred over another. For example, there are applications when it is necessary to have an *orthonormal basis* for the tangent plane – i.e., two unit vectors in the plane that are perpendicular to each other. For an orthogonal parameterization $\mathbf{r}(u, v)$, we need only rescale $\mathbf{r}_u$ and $\mathbf{r}_v$ into unit vectors to obtain the desired orthonormal basis (such rescaling is known as *normalization*).



However, if $\mathbf{r}(u,v) = \langle u, v, f(u,v) \rangle$, then $\mathbf{r}_u \cdot \mathbf{r}_v = f_u f_v$, which typically is non-zero. However, if we let $U(x,y,z) = z - f(x,y)$ where $x = u$, $y = v$, then the vector

$$\mathbf{w} = \nabla U \times \mathbf{r}_u$$

is in the tangent plane and is perpendicular to $\mathbf{r}_u$. Normalizing $\mathbf{r}_u$ and $\mathbf{w}$ resuts in an orthonormal basis for each tangent plane.



Conversely, proofs of theorems often begin with an assumption that a parametric or level surface implicitly defines z as a function $z = f(x,y)$ in a coordinate patch of the surface containing a given point, thus allowing techniques and ideas as in chapter 2 (or we may assume that x is implicitly defined as a function of y, z, etcetera).

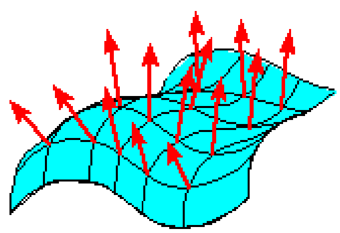
Check Your Reading: What is $\nabla U(1,2,-3)$ if we let $U(x,y,z) = f(x,y) - z$ in example 5?

Orientability and the Unit Surface Normal

Geometrically, saying that a level surface is *smooth* is equivalent to saying that it is *orientable*, where by orientable we mean that if we define the *unit surface* *normal* to be

$$\mathbf{n} = \frac{\nabla U}{||\nabla U||}$$

then $\mathbf{n}$ varies continuously across the surface. Intuitively, orientable means that if the initial points of the *unit surface normals* $\mathbf{n}$ are placed on the surface, then their terminal points are all on the same side of the surface.



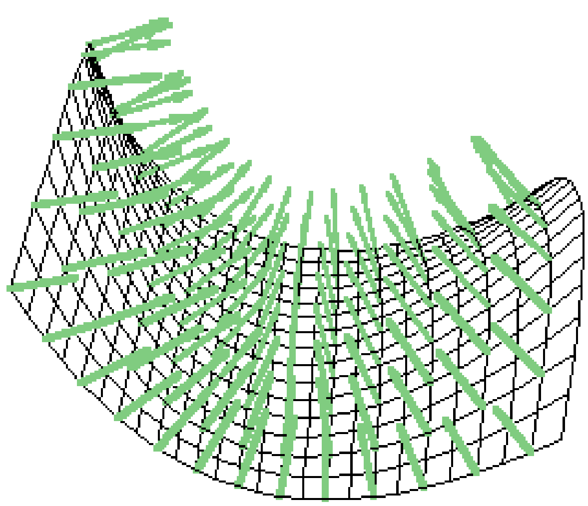
If a surface is closed (such as a sphere), then we assume its orientation to be with all normal vectors pointing toward the outside of the surface.

EXAMPLE 6 Explain why the surface $z = x^2 \cos(y)$ is orientable.

Solution: Since $z = x^2 \cos(y)$ is the same as $x^2 \cos(y) - z = 0$, we let $U(x, y, z) = z - x^2 \cos(y)$, so that

$$\nabla U = \left\langle -2x \cos(y), x^2 \sin(y), 1 \right\rangle$$

Since the z-component of ∇U is 1, it is not possible for ∇U to ever be 0. In fact, because the z-component of ∇U is positive, the normal vectors must all be *above* the surface $z = x^2 \cos(y)$.



It follows that a parametric surface is orientable if $\mathbf{n}(u,v)$ varies continuously across the surface *and* defines only one surface normal at each point on the surface.

EXAMPLE 7 Find the unit surface normal of the surface

$$\mathbf{r}(\theta,z)=\langle r\cos(\theta),r\sin(\theta),\cosh(r)\rangle$$

for θ in $[0,2\pi]$ and r in $[0,1]$ is orientable. Explain why the surface is orientable.

Solution: Since $\mathbf{r}_r=\langle\cos(\theta),\sin(\theta),\sinh(r)\rangle$ and $\mathbf{r}_\theta=\langle -r\sin(\theta),r\cos(\theta),0\rangle$, their cross product is

$$\begin{aligned}\mathbf{r}_r\times\mathbf{r}_\theta &= \langle\cos(\theta),\sin(\theta),\sinh(r)\rangle\times\langle -r\sin(\theta),r\cos(\theta),0\rangle\\ &= r\langle -\sinh(r)\cos(\theta),-\sinh(r)\sin(\theta),1\rangle\end{aligned}$$

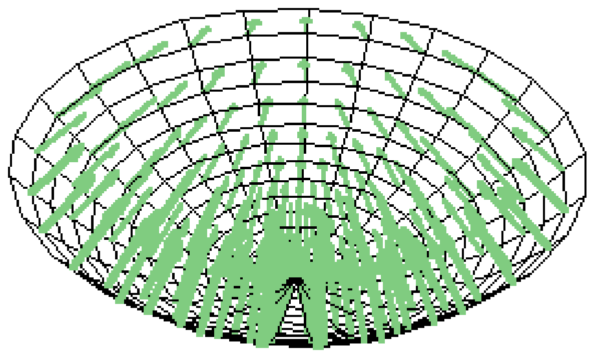
The square of the magnitude of their cross product is

$$\begin{aligned}\|\mathbf{r}_r\times\mathbf{r}_\theta\|^2 &= r^2\left(\sinh^2(r)\cos^2(\theta)+\sinh^2(r)\sin^2(\theta)+1\right)\\ &= r^2\left(\sinh^2(r)+1\right)\\ &= r^2\cosh^2(r)\end{aligned}$$

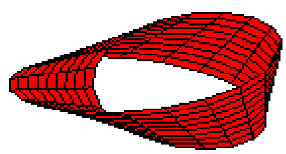
Thus, $\|\mathbf{r}_r\times\mathbf{r}_\theta\|=r\cosh(r)$, so that the surface unit normal is

$$\begin{aligned}\mathbf{n} &= \frac{\mathbf{r}_r\times\mathbf{r}_\theta}{\|\mathbf{r}_r\times\mathbf{r}_\theta\|}\\ &= \frac{r\langle -\sinh(r)\cos(\theta),-\sinh(r)\sin(\theta),1\rangle}{r\cosh(r)}\\ &= \left\langle\frac{-\sinh(r)}{\cosh(r)}\cos(\theta),\frac{-\sinh(r)}{\cosh(r)}\sin(\theta),\frac{1}{\cosh(r)}\right\rangle\\ &= \langle -\tanh(r)\cos(\theta),-\tanh(r)\sin(\theta),\operatorname{sech}(r)\rangle\end{aligned}$$

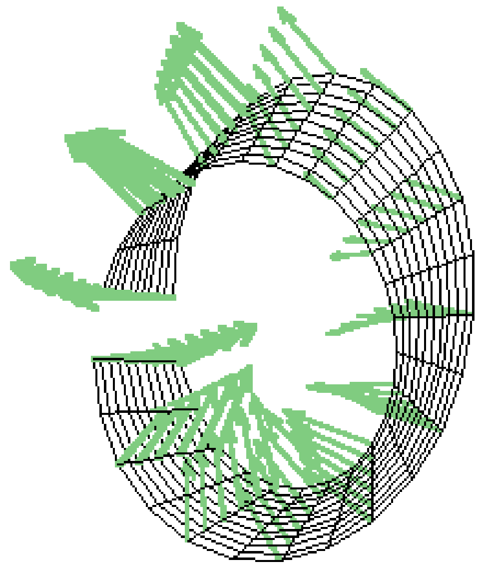
Since $\mathbf{n}(r,\theta)=\langle -\tanh(r)\cos(\theta),-\tanh(r)\sin(\theta),\operatorname{sech}(r)\rangle$ is continuous fro all r and θ, and since $\mathbf{n}(r,\theta)$ is 2π-periodic, the surface is orientable.



Not all surfaces can be oriented. For example, a Möbius strip, which can be formed by twisting a strip of paper one half turn and pasting the two ends together, cannot be oriented).



Indeed, if normal vectors are drawn on the surface, then the half-twist means that there will be a discontinuity in the assignment of unit normal vectors, or equivalently, a "sudden reversal" of the direction of the normals.



Exercises

Find the equation of the tangent plane to the given surface at the given point.

1. $x^2 + y^2 + z^2 = 11$ at $(1, 1, 3)$
2. $x^2 + y^2 + z^2 = 9$ at $(2, 1, 2)$
3. $xy + z^2 = 4$ at $(1, 2, 2)$
4. $x^2 y + z^2 = 4$ at $(1, 2, 2)$
5. $3x + 4y + 2z = 13$ at $(1, 2, 1)$
6. $3x - 2y + 4z = -4$ at $(2, 1, -2)$
7. $x^2 + y^2 - z^2 = 1$ at $(1, 1, 1)$
8. $x^2 - y^2 - z^2 = 2$ at $(2, 1, 1)$
9. $xe^y + z = 2$ at $(1, 0, 1)$
10. $\sin(xy) + z = 2$ at $(\pi, 1, 2)$

Find the equation of the tangent plane to $\mathbf{r}(u, v)$ *at the point* $\mathbf{r}(p, q)$ *for the given* (p, q).

11. $\mathbf{r} = \langle v \sin(u), v \cos(u), v \rangle$
 $(p, q) = (\pi/4, 2)$
12. $\mathbf{r} = \langle v \cos(u), v \sin(u), v \rangle$
 $(p, q) = (\pi/2, 1)$
13. $\mathbf{r} = \langle \cos(u), \sin(u), v \rangle$
 $(p, q) = (\pi/4, 3)$
14. $\mathbf{r} = \langle \cos(u), \sin(u), v \rangle$
 $(p, q) = (\pi/2, 1)$
15. $\mathbf{r} = \langle v \sin(u), v \cos(u), uv \rangle$
 $(p, q) = (\pi/3, 1)$
16. $\mathbf{r} = \left\langle v \sin(u), v^2, v \cos(u) \right\rangle$
 $(p, q) = (\pi/4, 1)$
17. $\mathbf{r} = \langle \sin(v) \sin(u), \cos(v) \sin(u), \cos(u) \rangle$
 $(p, q) = (\pi/4, \pi/4)$
18. $\mathbf{r} = \langle \sin(v) \sin(u), \cos(v) \sin(u), \cos(u) \rangle$
 $(p, q) = (\pi/3, \pi/4)$
19. $\mathbf{r} = \langle e^v \sin(u), e^v \cos(u), e^{-v} \rangle$
 $(p, q) = (\pi, 1)$
20. $\mathbf{r} = \langle \sin(u) \cosh(v), \sin(u) \sinh(v), \cos(u) \rangle$
 $(p, q) = (\pi, \ln 2)$

Find the tangent plane to the graph of $f(x, y)$ *at* $(p, q, f(p, q))$ *by (a) using methods from section 2-5, (b) considering the surface* $z - f(x, y) = 0$, *and (c)* *finding the tangent plane to a parameterization of the form*

$$\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle$$

The answer should be the same in all 3 cases.

21.	$f(x,y) = x^2 - y, \quad (p,q) = (2,1)$	22.	$f(x,y) = x^2 + y^2, \quad (p,q) = (1,1)$
23.	$f(x,y) = 3x + 2y + 1, \quad (p,q) = (0,0)$	24.	$f(x,y) = xy, \ (p,q) = (1,1)$
25.	$f(x,y) = xe^{xy}, \quad (p,q) = (2,0)$	26.	$f(x,y) = \ln\left(x^2 + y^2\right), \quad (p,q) = (1,1)$

27. Is the surface $\mathbf{r}(u,v) = \langle v\sin(u), v\cos(u), v\rangle$, u in $[0, 2\pi]$, v in $[1,2]$, orientable? Explain.

28. Consider the surface parameterized by

$$\mathbf{r}(u,v) = \left\langle v\cos\left(\frac{u}{2}\right), \sin(u), v\sin\left(\frac{u}{2}\right)\right\rangle$$

for u in $[0, 2\pi]$ and v in $[-0.3, 0.3]$. Find $\mathbf{r}_u$ and $\mathbf{r}_v$ and then calculate

$$\mathbf{r}_u(0,v), \quad \mathbf{r}_u(2\pi,0), \quad \mathbf{r}_v(0,v), \ \textit{and} \ \ \mathbf{r}_v(2\pi,0)$$

Use these to calculate

$$\mathbf{n}(0,v) = \frac{\mathbf{r}_u(0,v)\times\mathbf{r}_v(0,v)}{\|\mathbf{r}_u(0,v)\times\mathbf{r}_v(0,v)\|} \quad \textit{and} \quad \mathbf{n}(2\pi,v) = \frac{\mathbf{r}_u(2\pi,v)\times\mathbf{r}_v(2\pi,v)}{\|\mathbf{r}_u(2\pi,v)\times\mathbf{r}_v(2\pi,v)\|}$$

Is $\mathbf{n}(0,v) = \mathbf{n}(2\pi,v)$? Is the surface orientable?

29. Find the equation of the tangent plane to the cone $x^2 + y^2 = z^2$ at $(1,0,1)$ using (a) a level surface and (b) the parameterization

$$\mathbf{r}(u,v) = \langle v\cos(u), v\sin(u), v\rangle$$

at $\mathbf{r}(0,1)$. The answer should be the same in both cases.

30. Find the equation of the tangent plane to

$$\mathbf{r}(u,v) = \left\langle ue^{v}, ue^{-v}, u\right\rangle$$

at $\mathbf{r}(1,0)$. Then find the equation of the tangent plane again using a level surface representation of $\mathbf{r}(u,v)$. The answer should be the same in both cases.

31. Show that the equation of the tangent plane to an elliptic paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

at a point (m,n,p) on the elliptic paraboloid is of the form

$$\frac{z+p}{2c} = \frac{mx}{a^2} + \frac{ny}{b^2}$$

32. Find the equation of the tangent plane to $x^2 + y^2 = z^2$ at the point (m, n, p), and then show that it must pass through the line $mx + ny = 0$ in the xy-plane.

33. If (m, n, p) is a point on a sphere of radius R centered at the origin, what is the equation of the tangent plane to the sphere at (m, n, p)?

34. Show that the normal vector to a sphere of radius R at a point (m, n, p) is parallel to the radius vector $\langle m, n, p \rangle$. What does this say about the relationship between the radius vector $\langle m, n, p \rangle$ and any vector tangent to the sphere at (m, n, p)?

35. Show that any tangent plane to $z = x^2 - y^2$ intersects the surface in two perpendicular lines.

36. Show that any tangent plane to $x^2 + y^2 - z^2 = 1$ intersects the surface in two lines.

37. Determine the longitude θ_0 and latitude φ_0 of your present location. Assume that earth is a sphere with latitude-longitude parameterization of

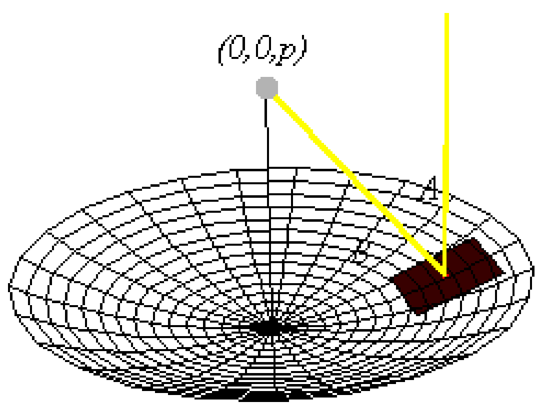
$$\mathbf{r}(\varphi, \theta) = \langle 3960 \cos(\varphi) \cos(\theta), 3960 \cos(\varphi) \sin(\theta), 3960 \sin(\varphi) \rangle$$

What is the equation of the tangent plane to the earth at your location? (**note:** this problem assumes an xyz-coordinate system at the center of the earth with z-axis through the poles and x-axis at 0° longitude).

38. A parabolic mirror is in the shape of a paraboloid with equation

$$4pz = x^2 + y^2$$

where p is a number. Show that a vertical line through a point (a, b, c) forms the same angle with the tangent plane at (a, b, c) as does the line through (a, b, c) and $(0, 0, p)$. (i.e., that a vertical ray of light is reflected by the paraboloid to the focus at $(0, 0, p)$).



39. Explain why if $\mathbf{n}$ denotes the surface normal of a surface $\mathbf{r}(u, v)$, then $\mathbf{n}(u, v)$ is a parameterization of a section of the unit sphere. What section of the unit sphere is parameterized by the surface normal **n** to the right circular cylinder

$$\mathbf{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$$

40. What section of the unit sphere is parameterized by the surface normal **n** to the surface

$$\mathbf{r}(u,v) = \langle v\cos(u), v\sin(u), \cosh(v) \rangle$$

for $v \geq 0$ and u in $[0, 2\pi]$. (see exercise 39).

41. Write to Learn: Write a short essay which explains why the tangent plane to a plane is the plane itself. In particular, demonstrate using a parametric representation of a plane

$$\mathbf{r}(u,v) = \langle a + mu + nv, b + pu + qv, c + su + tv \rangle$$

for constants $a, b, c, m, n, p, q, s,$ and t, demonstrate using a level surface representation

$$\alpha(x-a) + \beta(y-b) + \gamma(z-c) = 0$$

for α, β, and γ constant, and also demonstrate using a functional form by solving for z when $\gamma \neq 0$.

42. Write to Learn: Suppose that two smooth surfaces $F(x,y,z) = k$ and $G(x,y,z) = l$ both contain the point (x_0, y_0, z_0) and that $\nabla F(x_0, y_0, z_0) = c\nabla G(x_0, y_0, z_0)$ for some number c. Write a short essay which explains why the two surfaces are tangent at (x_0, y_0, z_0).