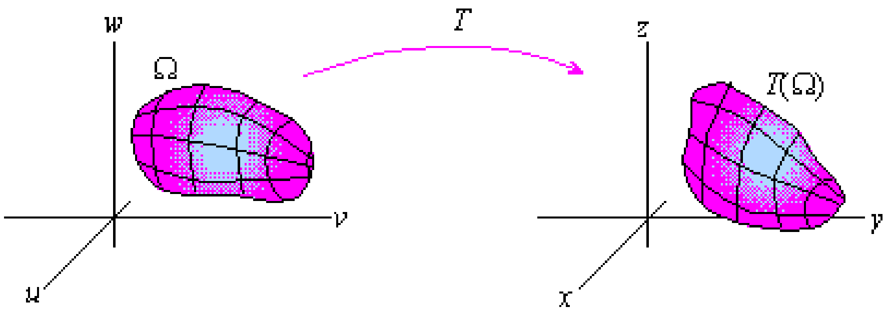
# Curvilinear Coordinates

## Cylindrical Coordinates

A *3-dimensional* coordinate transformation is a mapping of the form

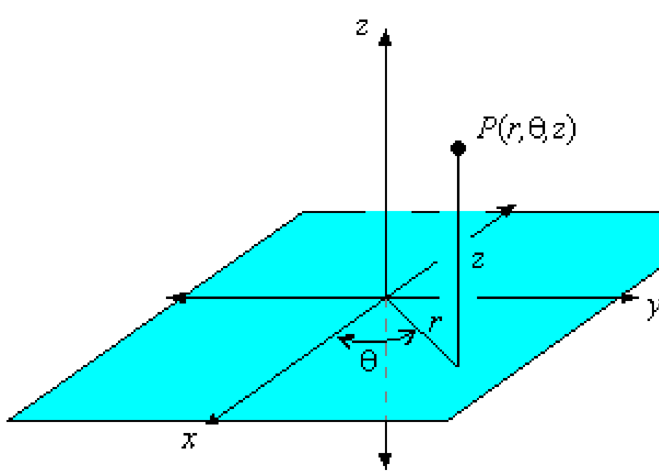
$$T(u,v,w) = \langle x(u,v,w), y(u,v,w), z(u,v,w) \rangle$$

Correspondingly, a 3-dimensional coordinate transformation $T$ maps a solid $\Omega$ in the $uvw$-coordinate system to a solid $T(\Omega)$ in the $xyz$-coordinate system (and similarly, $T$ maps curves in $uvw$ to curves in $xyz$, surfaces in $uvw$ to surfaces in $xyz$, and so on).




In this section, we introduce and explore two of the more important 3-dimensional coordinate transformations.

To begin with, the *cylindrical* coordinates of a point $P$ are Cartesian coordinates in which the $x$ and $y$ coordinates have been transformed into polar coordinates (and the $z$-coordinate is left as is).



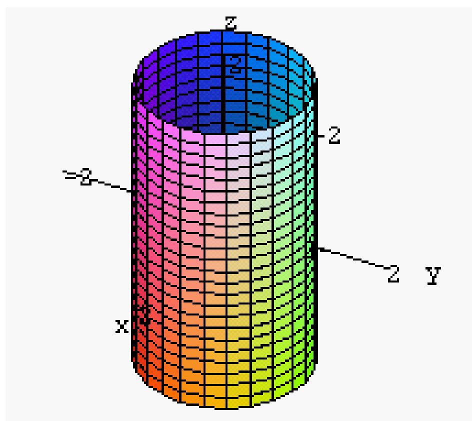
Not surprisingly, to convert to cylindrical coordinates, we simply apply $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to the $x$ and $y$ coordinates. That is, the cylindrical coordinate transformation is

$$T(r, \theta, z) = \langle r\cos(\theta), r\sin(\theta), z\rangle$$

Cylindrical coordinates get their name from the fact that the surface "$r =$ constant" is a cylinder. For example, the cylinder

$$\mathbf{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z\rangle$$

is obtained by setting $r = 1$ in the cylindrical coordinate transformation.




Likewise, parameterizations of many other level surfaces can be derived from the cylindrical coordinate transformation.

In particular, if points in the $xy$-plane are in polar coordinates, then $z = f(r, \theta)$ is a *surface* in 3 dimensional space, and the parameterization of that surface is

$$\mathbf{r}(r, \theta) = \langle \cos(\theta), \sin(\theta), f(r, \theta)\rangle$$

More generally, $U(r, \theta, z) = k$ defines a *level surface* in which the $xy$ components are represented in polar coordinates.

*EXAMPLE 1* Find a parametrization of the right circular cone

$$z^2 = x^2 + y^2$$

by pulling back into cylindrical coordinates.

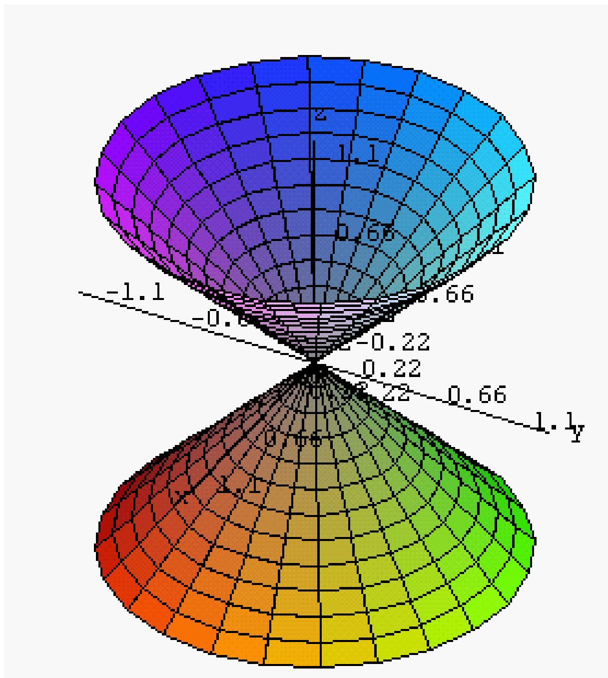
**Solution:** Transforming $x$ and $y$ into polar coordinates yields

$$z^2 = r^2, \quad z = r$$

Letting $z = r$ in the cylindrical coordinate transformation yields

$$\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), r\rangle$$

which is a parametrization of the right circular cone.



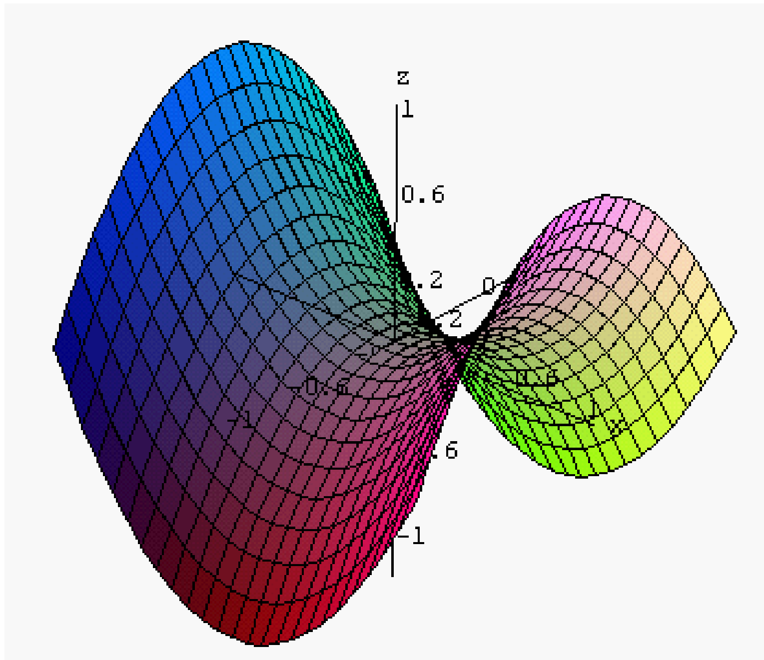

*EXAMPLE 2* Parameterize the surface $z = x^2 - y^2$ by pulling back into cylindrical coordinates

**Solution:** Setting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ leads to

$$z = r^2\cos^2(\theta) - r^2\sin^2(\theta) = r^2\cos(2\theta)$$

Thus, the parametrization is

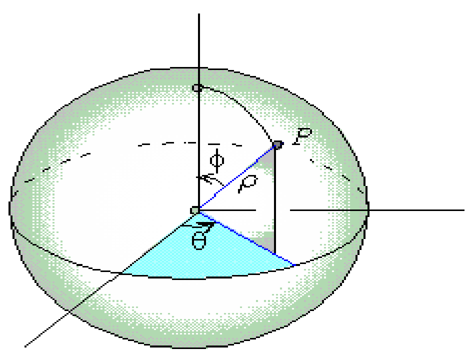
$$\mathbf{r}(r, \theta) = \left\langle r\cos(\theta), r\sin(\theta), r^2\cos(2\theta)\right\rangle$$



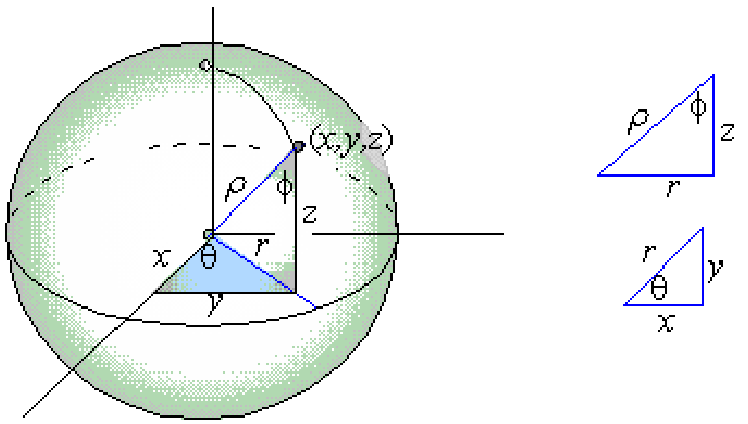

**Check your Reading:** In what plane are the cylindrical coordinates of a point the same as its polar coordinates?

**Spherical Coordinates**

The *spherical coordinates* of a point $P$ are defined to be $(\rho, \phi, \theta)$, where $\rho$ is the distance from $P$ to the origin, $\phi$ is the angle formed by the $z$-axis and the ray from the origin to $P$, and $\theta$ is the polar angle from polar coordinates.



Specifically, the Cartesian coordinates $(x, y, z)$ of a point $P$ are related to the spherical coordinates $(\rho, \phi, \theta)$ of $P$ through two right triangles. Relationships among $x$, $y$, $\theta$, and the polar distance $r$ are contained in the familiar polar coordinate triangle. Relationships among $r$, $z$, $\rho$, and $\phi$ are conveyed by a second right triangle.




These 2 triangles are at the heart of spherical coordinates. For example, the triangle imply the relationships

$$\begin{array}{ll} x = r\cos(\theta) & z = \rho\cos(\phi) \\ y = r\sin(\theta) & r = \rho\sin(\phi) \end{array} \tag{1}$$

so that if we eliminate $r$ using the fact that $r = \rho\sin(\phi)$, we obtain

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi) \tag{2}$$

which is the *coordinate transformation* that maps spherical coordinates into Cartesian coordinates.

*EXAMPLE 3* Transform the point $(4, \pi/3, \pi/2)$ from spherical into Cartesian coordinates.

**Solution:** The transformation (2) implies that

$$\begin{aligned} x &= 4\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) = 4\cdot\frac{\sqrt{3}}{2}\cdot 0 = 0 \\ y &= 4\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{2}\right) = 4\cdot\frac{\sqrt{3}}{2}\cdot 1 = 2\sqrt{3} \\ z &= 4\cos\left(\frac{\pi}{3}\right) = 2 \end{aligned}$$

Thus, $(4, \pi/3, \pi/2)$ in spherical coordinates is the same point as $\left(0, 2\sqrt{3}, 2\right)$ in Cartesian coordinates.

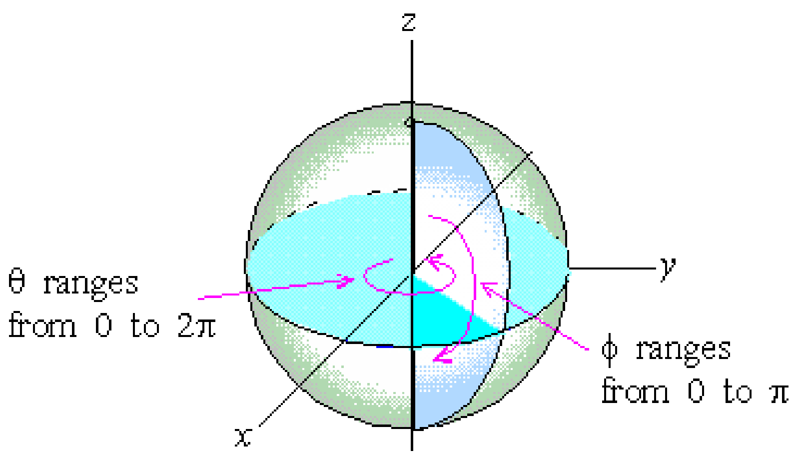
In spherical coordinates, $r = \rho \sin(\phi)$ and $z = \rho \cos(\phi)$, so that the polar $x^2 + y^2 = r^2$ becomes

$$x^2 + y^2 = \rho^2 \sin^2(\phi)$$

Moreover, $r^2 + z^2 = \rho^2$, so that we have the identity

$$x^2 + y^2 + z^2 = \rho^2 \tag{3}$$

Thus, if $R$ is constant, then $\rho = R$ is a sphere of radius $R$ centered at the origin. In addition, we usually restrict $\theta$ to $[0, 2\pi]$ and $\phi$ to $[0, \pi]$ so that the sphere is covered only once.

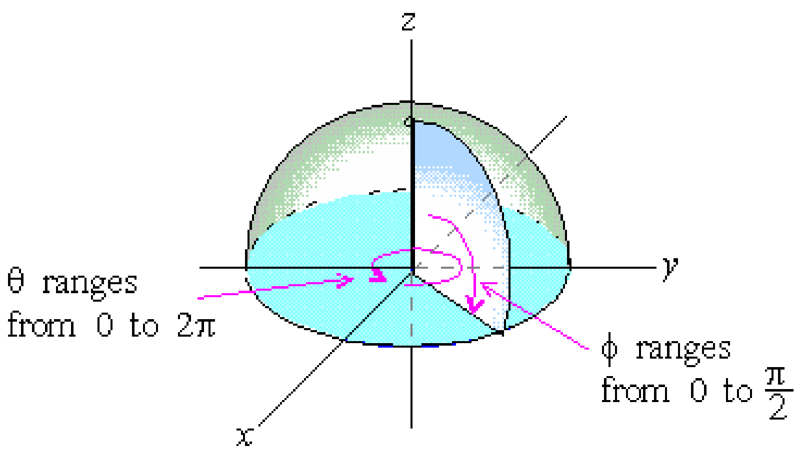



Restricting $\phi$ and $\theta$ to smaller intervals yields smaller sections of a sphere.

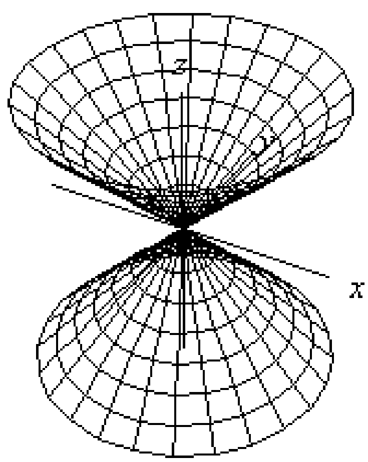
*EXAMPLE 4* What section of the sphere $\rho = 1$ is given by $\phi$ in $[0, \pi/2]$, $\theta$ in $[0, 2\pi]$?

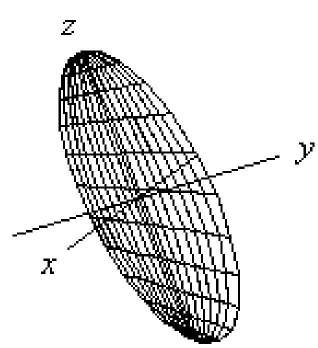
**Solution:** Since $\phi = \pi/2$ is the $xy$-plane, the set of points $\rho = 1$, $\phi$ in $[0, \pi/2]$, $\theta$ in $[0, 2\pi]$ is the part of the unit sphere above the

$xy$-plane—i.e., the upper hemisphere.




Similarly, $\phi = k$ for $k$ constant is a cone with sides at angle $k$ to the vertical, and $\theta = c$ for $c$ constant is a vertical plane of the form $y = \tan(c)\, x$




$\phi = k$ for $k$ constant $\qquad\qquad$ $\theta = c$ for $c$ constant

**Check your Reading:** For what values of $\phi, \theta$ is the *lower unit hemisphere* defined?

Since $x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$, the position vector of a point in space is

$$\mathbf{r}(\phi,\theta) = \langle \rho \sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\rangle = \rho\mathbf{e}_\rho(\phi,\theta)$$

where we define $\mathbf{e}_\rho(\phi,\theta) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$. That is, the surface $\mathbf{r}(\phi,\theta)$ can be written more compactly as

$$\mathbf{r}(\phi,\theta) = \rho\mathbf{e}_\rho(\phi,\theta)$$

It follows that the parameterization of the graph of $\rho = f(\phi,\theta)$ is given by

$$\mathbf{r}(\phi,\theta) = f(\phi,\theta)\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle \tag{4}$$

or equivalently, $\mathbf{r}(\phi,\theta) = f(\phi,\theta)\,\mathbf{e}_\rho(\phi,\theta)$

In particular, if we substitute $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$ into the equation of a level surface and *solve for* $\rho$, then (4) parameterizes a *coordinate patch* on that level surface.

*EXAMPLE 5* Pull back into spherical coordinates to obtain a parameterization of the hyperboloid in two sheets

$$z^2 - x^2 - y^2 = 1$$

**Solution:** Substituting from (2) and simplifying yields

$$\begin{aligned} \rho^2\cos^2(\phi) - \rho^2\sin^2(\phi)\cos^2(\theta) - \rho^2\sin^2(\phi)\sin^2(\theta) &= 1 \\ \rho^2\cos^2(\phi) - \rho^2\sin^2(\phi)\left[\cos^2(\theta) + \sin^2(\theta)\right] &= 1 \\ \rho^2\cos^2(\phi) - \rho^2\sin^2(\phi) &= 1 \end{aligned}$$

However, $\cos^2(\phi) - \sin^2(\phi) = \cos(2\phi)$, so that $\rho^2\cos(2\phi) = 1$ and

$$\rho^2 = \sec(2\phi) \tag{5}$$

Thus, the upper sheet of the hyperboloid is parameterized by $\mathbf{r}(\phi,\theta) = [\sec(2\phi)]^{1/2}\,\mathbf{e}_\rho$, which yields

$$\mathbf{r}(\phi,\theta) = \sqrt{\sec(2\phi)}\ \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$$

since $\mathbf{e}_\rho = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$.

Spherical coordinates provides us a straightforward means of defining certain types of surfaces of revolution. If $0 \leq \alpha \leq \pi$ and if $f(\phi) \geq 0$ on $[0,\alpha]$ does **not** depend on $\theta$, then the surface $\rho = f(\phi)$ on $[0,\alpha]$ is the revolution of the curve

$$\mathbf{r}(\phi,0) = \langle f(\phi)\sin(\phi), 0, f(\phi)\cos(\phi)\rangle, \ \ \phi \ in \ \ [0,\alpha]$$

in the $xz$-plane around the $z$-axis. Correspondingly, it has a parameterization of

$$\mathbf{r}(\phi,\theta) = f(\phi)\,\mathbf{e}_\rho(\phi,\theta)$$

If $\alpha = \pi$ and $f(\phi)$ is positive and continuous on $[0,\pi]$, then $\rho = f(\phi)$ is called a radially symmetric *deformation of the sphere*.

*EXAMPLE 6* Discuss the graph of the surface

$$\rho = 5 + 0.1\phi\sin(7\phi), \quad \phi \ in \ \ [0,\pi]$$

**Solution:** The parameterization is given by .

$$\mathbf{r}(\phi,\theta) = (5 + 0.1\phi\sin(7\phi))\,\mathbf{e}_\rho(\phi,\theta)$$

which by definition of $\mathbf{e}_\rho(\phi,\theta)$ leads to

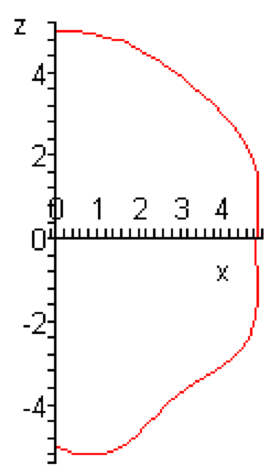
$$x = (5 + 0.1\phi\sin(7\phi))\sin(\phi)\cos(\theta), \quad y = (5 + 0.1\phi\sin(7\phi))\sin(\phi)\sin(\theta)$$

$$z = (5 + 0.1\phi\sin(7\phi))\cos(\phi)$$

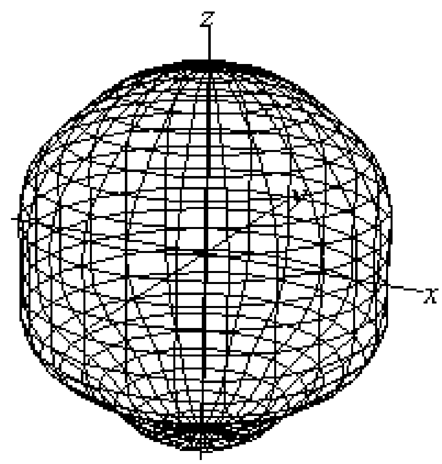
The surface is the revolution of the curve

$$\mathbf{r}(\phi,0) = \langle(5 + 0.1\phi\sin(7\phi))\sin(\phi), 0, (5 + 0.1\phi\sin(7\phi))\cos(\phi)\rangle$$

about the $z$-axis. The curve is shown below:



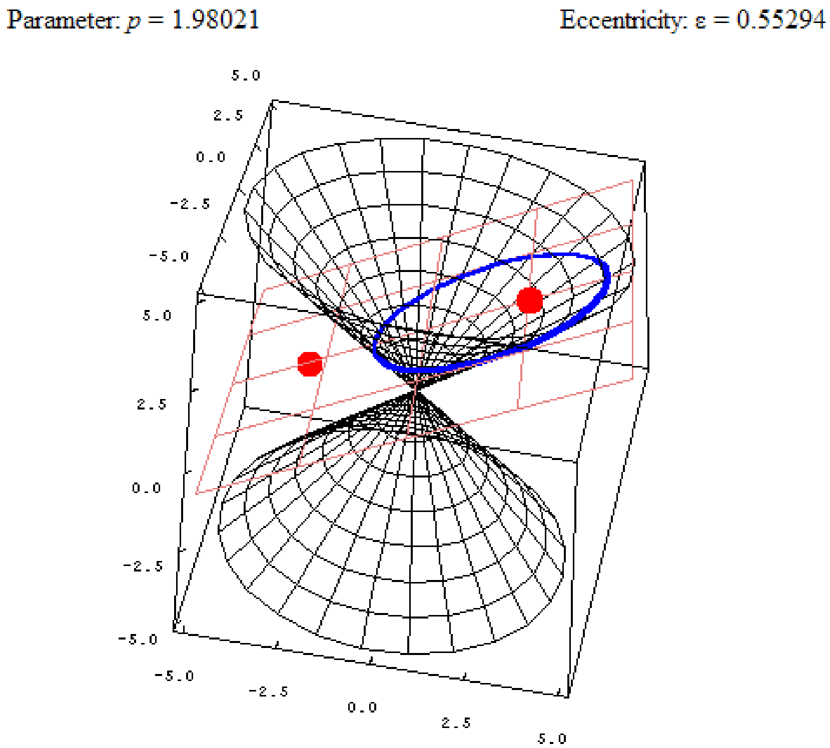
The resulting surface of revolution then follows:




**Check your Reading:** Is the surface in example 6 a deformation of the sphere?

## Conic Sections

A *conic section* is the curve formed by the intersection of a plane with the right circular cone $x^2 + y^2 = z^2$. For example, the plane in the conic explorer below is given by $z = p + \varepsilon x$, where $p$ is the *parameter* of the conic and $\varepsilon > 0$ is its *eccentricity:*




If $\varepsilon = 0$, then the conic is a circle. If $0 < \varepsilon < 1$, then the conic is an ellipse. If $\varepsilon = 1$, then the conic is a parabola, and if $\varepsilon > 1$, then the conic is a hyperbola.

In cylindrical coordinates, the plane is given by $z = p + \varepsilon r \cos(\theta)$ and the cone is given by $r^2 = z^2$. As a result, the intersection of the plane and the cone is given by

$$\begin{aligned} r &= p + \varepsilon r \cos(\theta) \\ r - \varepsilon r \cos(\theta) &= p \\ r\,(1 - \varepsilon \cos(\theta)) &= p \end{aligned}$$

which results in

$$r = \frac{p}{1 - \varepsilon \cos(\theta)} \tag{6}$$

This is actually the graph of the projection in polar coordinates of the conic into the $xy$-plane.

*projectino*

Moreover, $z = \pm r$ and cylindrical coordinates implies that parameterization of the conic itself is

$$\boldsymbol{\rho}(\theta) = \left\langle \frac{p\cos(\theta)}{1 - \varepsilon\cos(\theta)}, \frac{p\sin(\theta)}{1 - \varepsilon\cos(\theta)}, \frac{p}{1 - \varepsilon\cos(\theta)} \right\rangle$$

where we use the vector-valued function $\boldsymbol{\rho}$ since its length is the spherical coordinate distance $\rho$.

*EXAMPLE 7* Find the projection of the conic with eccentricity $\varepsilon = 1$ and parameter $p = 4$. What type of conic is it? What is the parameterization of the conic itself?

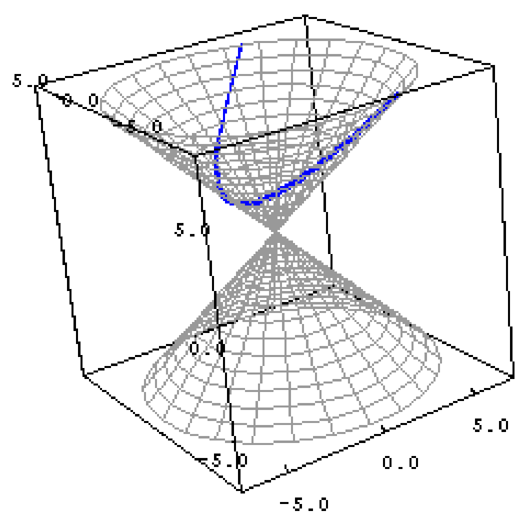
**Solution:** The projection is given by (6) with $\varepsilon = 1$ and $p = 4$:

$$r = \frac{4}{1 - \cos(\theta)}$$

This is a parabola with parameter $p = 4$. It is parameterized by

$$\boldsymbol{\rho}(t) = \left\langle \frac{4\cos(\theta)}{1 - \cos(\theta)}, \frac{4\sin(\theta)}{1 - \cos(\theta)}, \frac{4}{1 - \cos(\theta)} \right\rangle$$

which is shown below:




Conic projections of the form (6) are symmetric about the $x$-axis. Arbitrary conic projections follow from the intersection of the cone with an arbitrary plane, which is given by $z = ax + by + p$ with $a, b$, and $p$ constant.

*EXAMPLE 8* Find the projection of the conic formed by the intersection of $z = 3 + 0.5y$ with the right circular cone. What type of conic is it? What is the parameterization of the conic itself?

**Solution:** The intersection of $z = 3 + 0.5y$ with the right circular cone $r^2 = z^2$ is given by

$$r = 3 + 0.5r\sin(\theta) \qquad \Longrightarrow \qquad r - r0.5\sin(\theta) = 3$$

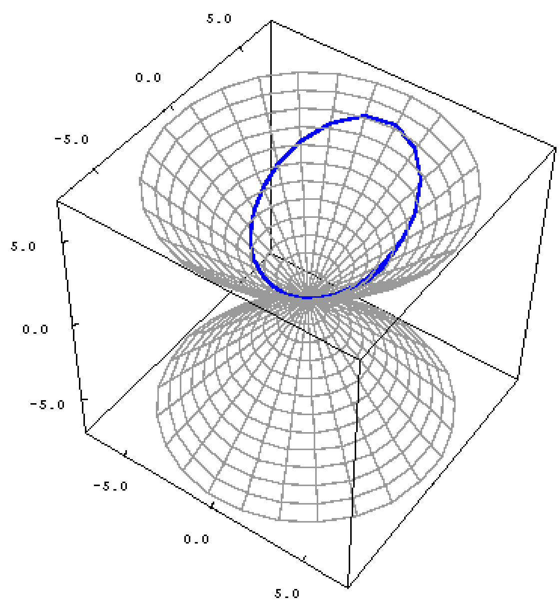
which results in the projection

$$r = \frac{3}{1 - 0.5\sin(\theta)}$$

This is an ellipse with parameter $p = 3$ and eccentricity $\varepsilon = 0.5$ that is symmetric about the $y$-axis. Moreover, the conic itself has the parameterization

$$\boldsymbol{\rho}(\theta) = \left\langle \frac{3\cos(\theta)}{1 - 0.5\sin(\theta)}, \frac{3\sin(\theta)}{1 - 0.5\sin(\theta)}, \frac{3}{1 - 0.5\sin(\theta)} \right\rangle$$

which is shown below:



## Exercises

**1.** Convert the following points from cylindrical coordinates $(r, \theta, z)$ to Cartesian coordinates (i.e., $xyz$ coordinates):

| | | | |
|---|---|---|---|
| a. | $(3, \pi/3, 3)$ | b. | $(7, \pi/2, 0)$ |
| c. | $(5, 0, 0)$ | d. | $(4, \pi, -2)$ |

**2.** What section of the cylinder $x^2 + y^2 = 1$ corresponds to cylindrical coordinates in the range $\theta$ in $[0, \pi]$ and $z$ in $[-1, 1]$?

**3.** Convert the following points from spherical coordinates $(\rho, \phi, \theta)$ to Cartesian coordinates:

| | | | |
|---|---|---|---|
| a. | $(3, \pi/3, \pi)$ | b. | $(7, \pi/2, \pi/4)$ |
| c. | $(-1, -\pi/2, 7\pi)$ | d. | $(5, 0, 0)$ |

**4.** What section of the unit sphere corresponds to spherical coordinates in the range $\phi$ in $[0, \pi]$ and $\theta$ in $[0, \pi]$?

*Find the pullback of the following surfaces into cylindrical coordinates. What is a parameterization of the surface?*

| | | | |
|---|---|---|---|
| 5. | $x^2 + y^2 = 25$ | 6. | $x^2 + y^2 = 30$ |
| 7. | $x^2 + y^2 - z^2 = 1$ | 8. | $x^2 - y^2 + z^2 = 9$ |
| 9. | $3x + 4y = 2$ | 10. | $x^2 + z^2 = 11$ |
| 11. | $x^2 + y^2 = z^2$ | 12. | $z = x^2 - y^2$ |

*Find the pullback of the following surfaces into spherical coordinates. What is a parameterization of the surface?*

| | | | |
|---|---|---|---|
| 13. | $x^2 + y^2 + z^2 = 25$ | 14. | $x^2 + y^2 + z^2 = 30$ |
| 15. | $x = 1$ | 16. | $x + y = 1$ |
| 17. | $x^2 + y^2 - z^2 = 1$ | 18. | $x^2 - y^2 + z^2 = 9$ |
| 19. | $z = 1 - 2y$ | 20. | $x^2 + z^2 = 11$ |
| 21. | $x^2 + y^2 = z^2$ | 22. | $x + y = 1$ |
| 23. | $x^2 z + y^2 z = 2xy$ | 24. | $z = x^2 - y^2$ |

*Find a parameterization of the conic section formed by the intersection of $z = p + \varepsilon x$ and the right circular cone. Then sketch its graph.*

| | | | |
|---|---|---|---|
| 25. | $p = 1, \quad \varepsilon = \frac{1}{2}$ | 26. | $p = 1, \quad \varepsilon = 0$ |
| 27. | $p = 2, \quad \varepsilon = 1$ | 28. | $p = -1, \quad \varepsilon = 0.1$ |
| 29. | $p = 1, \quad \varepsilon = 2$ | 30. | $p = 0, \quad \varepsilon = 1$ |

**31.** Discuss the surface of revolution given by

$$\rho = \frac{\phi}{\pi} + 1, \quad \phi \textit{ in } [0, \pi]$$

What is its parameterization? Is it a deformation of the sphere?

**32.** Discuss the surface of revolution given by

$$\rho = 2\sin(\phi), \quad \phi \ in \ [0, \pi]$$

What is its parameterization? Is it a deformation of the sphere?

**33.** The curve formed by the intersection of a sphere centered at the origin and a plane through the origin is called a *great circle.* Let's use spherical coordinates to develop a method for parameterizing a great circle.

1. (a) A non-vertical plane through the origin is of the form $z = ax + by$, where $a$ and $b$ are constants. Show that spherical coordinates transforms the equation into

$$\cos(\phi) = \sin(\phi)\left[a\cos(\theta) + b\sin(\theta)\right]$$

(b) Show that intersection of the plane with a sphere of radius $R$ results in the parameterization

$$\mathbf{r}(t) = R\sin(\phi)\left\langle \cos(\theta), \sin(\theta), a\cos(\theta) + b\sin(\theta)\right\rangle$$

where $\tan(\phi) = a\cos(\theta) + b\sin(\theta)$. Then use a right triangle to find $\sin(\phi)$ in terms of $a\cos(\theta) + b\sin(\theta)$ to finish the parameterization.

**34.** Show that cylindrical coordinates results in the same parameterization for a great circle (see exercise 33) as does spherical coordinates.

**35.** If a point has a location of $(\rho, \phi, \theta)$ in spherical coordinates, then its longitude is $\theta$ and its latitude is

$$\varphi = \frac{\pi}{2} - \phi$$

What is the parameterization of a sphere of radius $R$ in latitude-longitude coordinates? (be sure to simplify expressions like

$$\sin\left(\frac{\pi}{2} - \varphi\right) \quad and \quad \cos\left(\frac{\pi}{2} - \varphi\right)$$

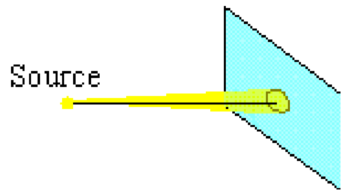
**36.** What is the equation in spherical coordinates of the sphere of radius $R$ centered at $(0, 0, R)$? What is its parameterization?

**37.** For $A$, $\omega$, $k$, and $\delta$ constants, the function

$$f(\rho, t) = \frac{A}{\rho}\cos(\omega t - k\rho + \delta)$$

is a *spherical wave* about the origin with angular frequency $\omega$, wavenumber $k$, and phase $\delta$. Explain why the spherical wave is the same in all directions. What happens to the spherical wave as the spherical distance $\rho$ goes to infinity?

**38. (Continues 37)** Spherical waves are often studied as they impact a small region of a plane.




In particular, for $R > 0$ constant, points $(x, y, R)$ in the plane $z = R$ are at a distance $\rho$ from the origin, where

$$\rho = \left(r^2 + R^2\right)^{1/2}$$

and $r^2 = x^2 + y^2$. Show that the Maclaurin's series of $\rho$ as a function of $r$ is of the form

$$\rho = R + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \frac{r^6}{16R^5} + \ldots$$

so that if $r^2 << R$ (i.e., if $r^2$ is much, much, less than $R$, corresponding to the region of impact in the $z = R$ plane being relatively small with respect to the distance from the source), then

$$\rho \approx R + \frac{r^2}{2R}$$

Finally, explain why for small regions in the $z = R$ plane, a spherical wave about the origin is approximately the same as

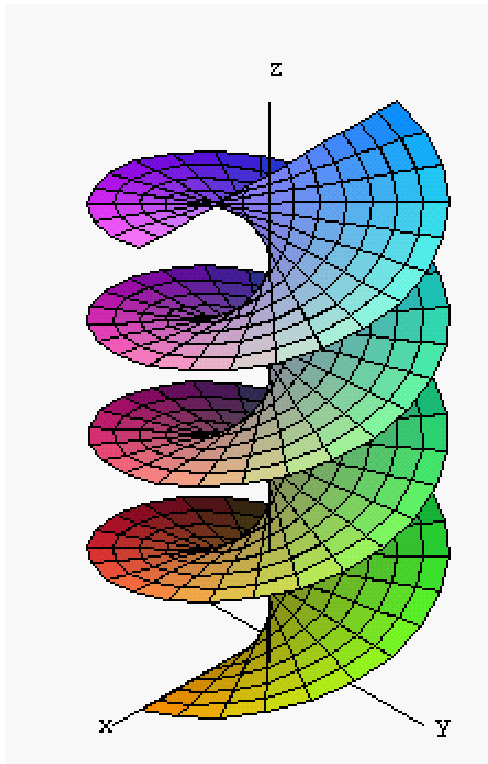
$$f(r, t) = \frac{A}{R} \cos\left(\omega t - \frac{kr^2}{2R} + \delta_1\right)$$

where $\delta_1 = \delta - kR$ is a new phase constant for the spherical wave. (i.e., a change in phase).

**39.** The interconnected double helix structure of DNA describes a *helicoid,* which is the surface parametrized by

$$\mathbf{r}(\theta, v) = \langle v\cos(\theta), v\sin(\theta), \theta\rangle$$

Show that $v = \theta\tan(\phi)$ and that the helicoid in spherical coordinates is given by $\rho = \theta \sec(\phi)$ .




**40.** What is the parameterization of the helicoid in exercise 39 in cylindrical coordinates?

**41. Write to Learn:** Explain in a short essay why a torus is radially symmetric but why there *does not exist* a postive continuous function $f(\phi)$ on $[0, \pi]$ such that the torus is given by $\rho = f(\phi)$ for $\phi$ in $[0, \pi]$ in spherical coordinates. (i.e., explain why it is impossible to deform a sphere into a torus).

**42.** Is an ellipsoid a radially symmetric deformation of the sphere? What about a paraboloid? Or a hyperboloid? Explain.

*Cylindrical and spherical coordinates are examples of curvilinear coordinate systems. Exercises 43-46 explore some additional curvilinear coordinates.*

**43. Ellcylindrical coordinates** assign a point $P$ in three dimensional space the coordinates $(u, \theta, z)$ , where

$$x = \cosh(u)\cos(\theta), \quad y = \sinh(u)\sin(\theta)$$

and where $z$ is the usual $z$-coordinate. What surface corresponds to $u = a$ for $a$ constant? What surface corresponds to $\theta = c$ for $c$ constant?

**44. Paraboloidal coordinates** assign a point $P$ in three dimensional space the coordinates $(u, v, \theta)$, where

$$x = uv\cos(\theta), \quad y = uv\sin(\theta), \quad z = \frac{u^2 - v^2}{2}$$

What surface corresponds to $u = a$ for $a$ constant? What surface corresponds to $v = b$ for $b$ constant? What surface corresponds to $\theta = c$ for $c$ constant?

**45. Bispherical coordinates** assign a point $P$ in three dimensional space the coordinates $(v, \phi, \theta)$, where

$$x = \frac{\sin(\phi)\cos(\theta)}{\cosh(v) - \cos(\phi)}, \quad y = \frac{\sin(\phi)\sin(\theta)}{\cosh(v) - \cos(\phi)}, \quad z = \frac{\sinh(v)}{\cosh(v) - \cos(\phi)}$$

Explain a surface of the form $v = k$ for $k$ constant is a sphere of radius $\operatorname{csch}(k)$ centered at $(0, 0, \coth(k))$.

**46. Toroidal coordinates** assign a point $P$ in three dimensional space the coordinates $(v, \phi, \theta)$, where

$$x = \frac{\sinh(v)\cos(\theta)}{\cosh(v) - \cos(\phi)}, \quad y = \frac{\sinh(v)\sin(\theta)}{\cosh(v) - \cos(\phi)}, \quad z = \frac{\sin(\phi)}{\cosh(v) - \cos(\phi)}$$

Explain a surface of the form $v = k$ for $k$ constant is a *torus,* and explain why if $k_1 \neq k_2$, then $v = k_1$ does not intersect $v = k_2$.