The Jacobian

The Jacobian of a Transformation

In this section, we explore the concept of a "derivative" of a coordinate transformation, which is known as the *Jacobian* of the transformation. However, in this course, it is the *determinant* of the Jacobian that will be used most frequently.

If we let $\mathbf{u} = \langle u, v \rangle$, $\mathbf{p} = \langle p, q \rangle$, and $\mathbf{x} = \langle x, y \rangle$, then $(x, y) = T(u, v)$ is given in vector notation by

$$\mathbf{x} = T(\mathbf{u})$$

This notation allows us to extend the concept of a total derivative to the total derivative of a coordinate transformation.

Definition 5.1: A coordinate transformation $T(\mathbf{u})$ is *differentiable* at a point $\mathbf{p}$ if there exists a matrix $J(\mathbf{p})$ for which

$$\lim_{\mathbf{u} \to \mathbf{p}} \frac{||T(\mathbf{u}) - T(\mathbf{p}) - J(\mathbf{p})(\mathbf{u} - \mathbf{p})||}{||\mathbf{u} - \mathbf{p}||} = 0 \qquad (1)$$

When it exists, $J(\mathbf{p})$ is the *total derivative* of $T(\mathbf{u})$ at $\mathbf{p}$.

In non-vector notation, definition 5.1 says that the total derivative at a point (p, q) of a coordinate transformation $T(u, v)$ is a matrix $J(u, v)$ evaluated at (p, q). In a manner analogous to that in section 2-5, it can be shown that this matrix is given by

$$J(u, v) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

(see exercise 46). The total derivative is also known as the *Jacobian Matrix* of the transformation $T(u, v)$.

EXAMPLE 1 What is the Jacobian matrix for the polar coordinate transformation?

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the Jacobian matrix is

$$J(r, \theta) = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

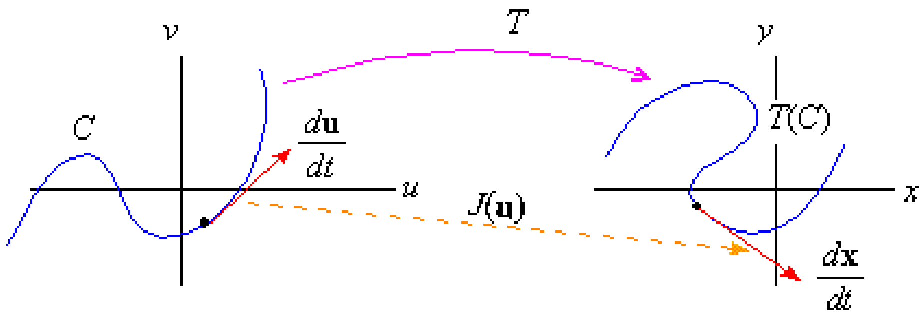
If $\mathbf{u}(t) = \langle u(t), v(t) \rangle$ is a curve in the uv-plane, then $\mathbf{x}(t) = T(u(t), v(t))$ is the image of $\mathbf{u}(t)$ in the xy-plane. Moreover,

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} x_u \frac{du}{dt} + x_v \frac{dv}{dt} \\ y_u \frac{du}{dt} + y_v \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}$$

The last vector is $d\mathbf{u}/dt$. Thus, we have shown that if $\mathbf{x}(t) = T(\mathbf{u}(t))$, then

$$\frac{d\mathbf{x}}{dt} = J(\mathbf{u}) \frac{d\mathbf{u}}{dt}$$

That is, the Jacobian maps tangent vectors to curves in the uv-plane to tangent vectors to curves in the xy-plane.



In general, the Jacobian maps any tangent vector to a curve at a given point to a tangent vector to the image of the curve at the image of the point.

EXAMPLE 2 Let $T(u,v) = \left\langle u^2 - v^2, 2uv \right\rangle$

a) Find the velocity of $\mathbf{u}(t) = \left\langle t, t^2 \right\rangle$ when $t = 1$.

b) Find the Jacobian and apply it to the vector in a)

c) Find $\mathbf{x}(t) = T(\mathbf{u}(t))$ in the xy-plane and then find its velocity vector at $t = 1$. Compare to the result in (b).

Solution: a) Since $\mathbf{u}'(t) = \langle 1, 2t \rangle$, the velocity at $t = 1$ is $\mathbf{u}'(1) = \langle 1, 2 \rangle$.

b) Since $x(u,v) = u^2 - v^2$ and $y(u,v) = 2uv$, the Jacobian of $T(u,v)$ is

$$J(u,v) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

Since $\mathbf{u}' = \langle 1, 2t \rangle$, we have

$$\begin{aligned} J(u,v)\,\mathbf{u}' &= \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} \begin{bmatrix} 1 \\ 2t \end{bmatrix} \\ &= \begin{bmatrix} 2u(1) - 2v(2t) \\ 2v(1) + 2u(2t) \end{bmatrix} \\ &= \begin{bmatrix} 2u - 4tv \\ 2v + 4tu \end{bmatrix} \end{aligned}$$

Substituting $\langle u,v\rangle = \left\langle t,t^2\right\rangle$ yields

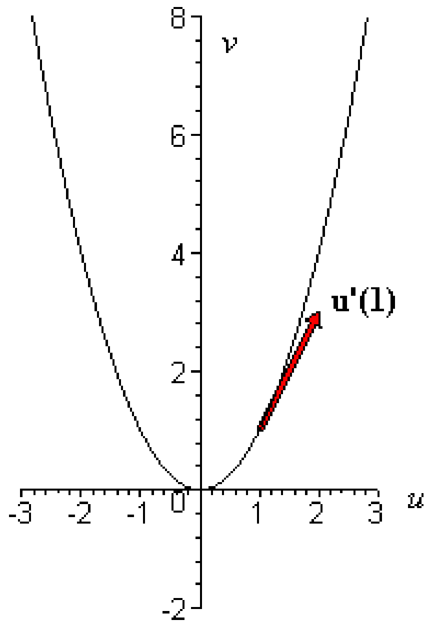
$$\mathbf{x}' = J(u,v)\,\mathbf{u}' = \begin{bmatrix} 2t-4t\left(t^2\right) \\ 2t^2+4t\,(t) \end{bmatrix} = \begin{bmatrix} 2t-4t^3 \\ 6t^2 \end{bmatrix}$$

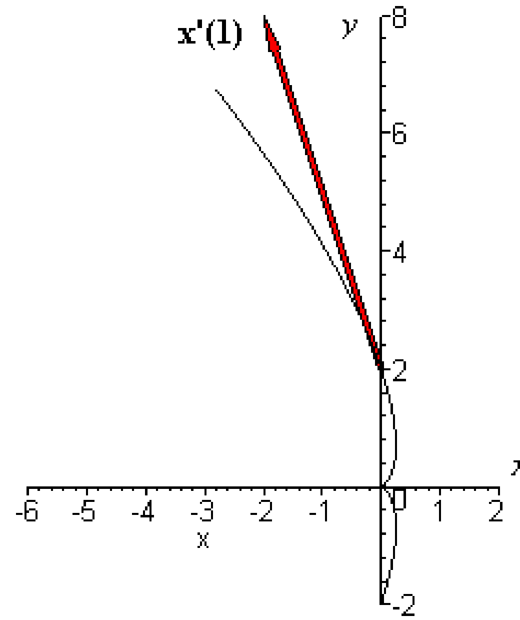
In vector form, $\mathbf{x}'(t) = \left\langle 2t-4t^3, 6t^2\right\rangle$, so that $\mathbf{x}'(1) = \langle -2,6\rangle$.

c) Substituting $u=t,\ v=t^2$ into $T(u,v) = \left\langle u^2-v^2, 2uv\right\rangle$ results in

$$\mathbf{x}(t) = \left(t^2-t^4, 2t^3\right)$$

which has a velocity of $\mathbf{x}'(t) = \left\langle 2t-4t^3, 6t^2\right\rangle$. Moreover, $\mathbf{x}'(1) = \langle -2,6\rangle$.





Check your Reading: At what point in the xy-plane is $\mathbf{x}'(1)$ tangent to the curve?

The Jacobian Determinant

The determinant of the Jacobian matrix of a transformation is given by

$$\det(J) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

However, we often use a notation for $\det(J)$ that is more suggestive of how the determinant is calculated.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The remainder of this section explores the Jacobian determinant and some of its more important properties.

EXAMPLE 3 Calculate the Jacobian Determinant of

$$T(u,v) = \left\langle u^2 - v, u^2 + v \right\rangle$$

Solution: If we identify $x = u^2 - v$ and $y = u^2 + v$, then

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (2u)(1) - (-1)(2u) \\ &= 4u \end{aligned}$$

Before we consider applications of the Jacobian determinant, let's develop some of the its properties. To begin with, if $x(u,v)$ and $y(u,v)$ are differentiable functions, then

$$\begin{aligned} \frac{\partial(y,x)}{\partial(u,v)} &= \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u} \\ &= -\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \\ &= -\frac{\partial(x,y)}{\partial(u,v)} \end{aligned}$$

from which it follows immediately that

$$\frac{\partial(x,x)}{\partial(u,v)} = \frac{\partial(y,y)}{\partial(u,v)} = 0$$

Similarly, if $f(u,v)$, $g(u,v)$, and $h(u,v)$ are differentiable, then

$$\begin{aligned} \frac{\partial(f+g,h)}{\partial(u,v)} &= \frac{\partial(f+g)}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial(f+g)}{\partial v}\frac{\partial h}{\partial u} \\ &= \frac{\partial f}{\partial u}\frac{\partial h}{\partial v} + \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} \\ &= \left(\frac{\partial f}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u}\right) + \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right) \\ &= \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(g,h)}{\partial(u,v)} \end{aligned}$$

The remaining properties in the next theorem can be obtained in similar fashion.

Theorem 5.2: If $f(u,v)$, $g(u,v)$, and $h(u,v)$ are differentiable functions and k is a number, then

$$\begin{array}{ll} \frac{\partial(g,f)}{\partial(u,v)} = -\frac{\partial(f,g)}{\partial(u,v)} & \frac{\partial(f+g,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(g,h)}{\partial(u,v)} \\ \frac{\partial(f,f)}{\partial(u,v)} = 0 & \frac{\partial(f-g,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(g,h)}{\partial(u,v)} \\ \frac{\partial(kf,g)}{\partial(u,v)} = k\frac{\partial(f,g)}{\partial(u,v)} & \frac{\partial(fg,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)}\, g + f\, \frac{\partial(g,h)}{\partial(u,v)} \end{array}$$

These and additional properties will be explored in the exercises.

EXAMPLE 4 Verify the property

$$\frac{\partial(fg,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)}\, g + f\, \frac{\partial(g,h)}{\partial(u,v)}$$

Solution: Direct calculation leads to

$$\begin{aligned} \frac{\partial(fg,h)}{\partial(u,v)} &= \frac{\partial(fg)}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial(fg)}{\partial v}\frac{\partial h}{\partial u} \\ &= \left(\frac{\partial f}{\partial u}\, g + f\, \frac{\partial g}{\partial u}\right)\frac{\partial h}{\partial v} - \left(\frac{\partial f}{\partial v}\, g + f\, \frac{\partial g}{\partial v}\right)\frac{\partial h}{\partial u} \\ &= \left(\frac{\partial f}{\partial u}\, \frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u}\right) g + f\left(\frac{\partial g}{\partial u}\, \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right) \\ &= \frac{\partial(f,h)}{\partial(u,v)}\, g + f\, \frac{\partial(g,h)}{\partial(u,v)} \end{aligned}$$

Check Your Reading: If k is constant and $f(u,v)$ is differentiable, then what is

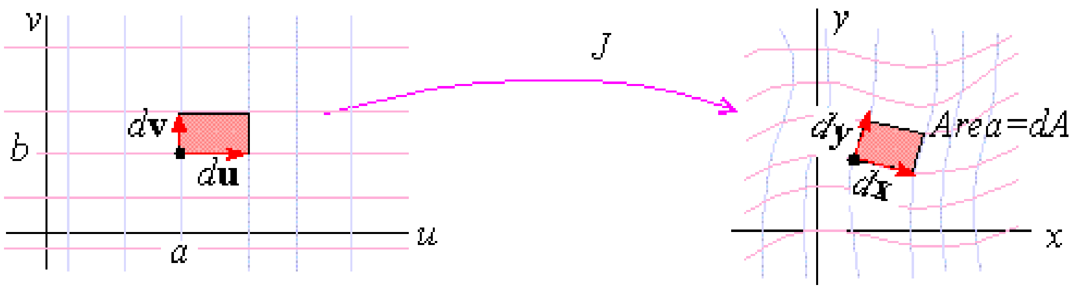
$$\frac{\partial(k,f)}{\partial(u,v)}?$$

The Area Differential

Let $T(u,v)$ be a smooth coordinate transformation with Jacobian $J(u,v)$, and let R be the rectangle spanned by $\mathbf{du} = \langle du, 0\rangle$ and $\mathbf{dv} = \langle 0, dv\rangle$. If du and dv are sufficiently close to 0, then $T(R)$ is approximately the same as the parallelogram spanned by

$$\begin{aligned} d\mathbf{x} &= J(u,v)\, d\mathbf{u} = \langle x_u du, y_u du, 0\rangle \\ d\mathbf{y} &= J(u,v)\, d\mathbf{v} = \langle x_v dv, y_v dv, 0\rangle \end{aligned}$$

If we let dA denote the area of the parallelogram spanned by $d\mathbf{x}$ and $d\mathbf{y}$, then dA approximates the area of $T(R)$ for du and dv sufficiently close to 0.



The cross product of $d\mathbf{x}$ and $d\mathbf{y}$ is given by

$$d\mathbf{x} \times d\mathbf{y} = \left\langle 0, 0, \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\rangle dudv$$

from which it follows that

$$dA = ||d\mathbf{x} \times d\mathbf{y}|| = |x_u y_v - x_v y_u|\, dudv \tag{2}$$

Consequently, the *area differential* dA is given by

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{3}$$

That is, the area of a small region in the uv-plane is scaled by the Jacobian determinant to approximate areas of small images in the xy-plane.

EXAMPLE 5 Find the Jacobian determinant and the area differential of $T(u,v) = \left\langle u^2 - v^2, 2uv \right\rangle$ at $(u,v) = (1,1)$, What is the approximate area of the image of the rectangle $[1, 1.4] \times [1, 1.2]$?

Solution: The Jacobian determinant is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (2u)(2u) - (-2v)(2v) \\ &= 4u^2 + 4v^2 \end{aligned}$$

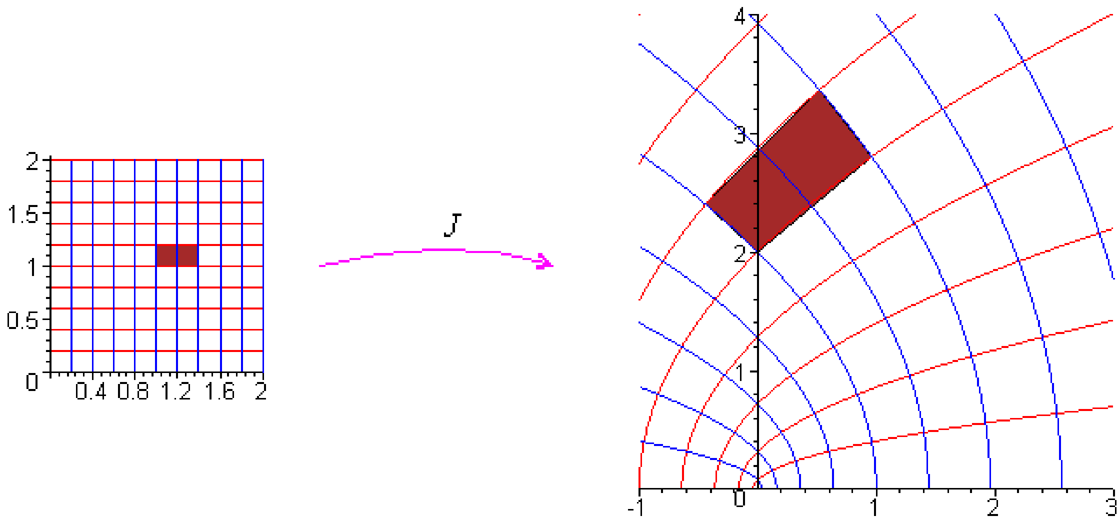
Thus, the area differential is given by

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv = \left(4u^2 + 4v^2\right) dudv$$

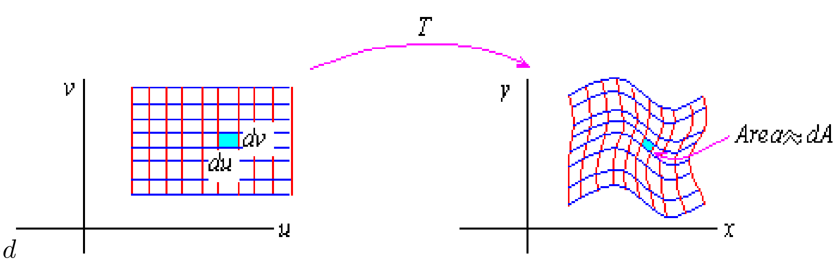
On the rectangle $[1, 1.4] \times [1, 1.2]$, the variable u changes by $du = 0.4$ and v changes by $dv = 0.2$. We evaluate the Jacobian at $(u, v) = (1, 1)$ and obtain the area

$$dA = (4 \cdot 1^2 + 4 \cdot 1^2) \cdot 0.4 \cdot 0.2 = 0.32$$

which is the approximate area in the xy-plane of the image of $[1, 1.4] \times [1, 1.2]$ under $T(u, v)$.



Let's look at another interpretation of the area differential. If the coordinate curves under a transformation $T(u, v)$ are sufficiently close together, then they form a grid of lines that are "practically straight" over short distances. As a result, sufficiently small rectangles in the uv-plane are mapped to small regions in the xy-plane that are practically the same as parallelograms.



11−6−d

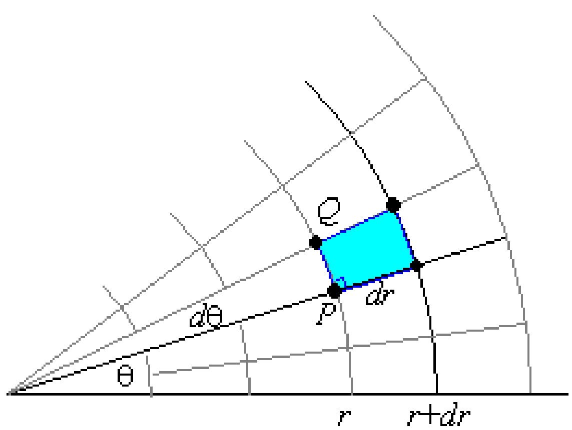
Consequently, the area differential dA approximates the area in the xy-plane of the image of a rectangle in the uv-plane as long as the rectangle in the uv-plane is sufficiently small.

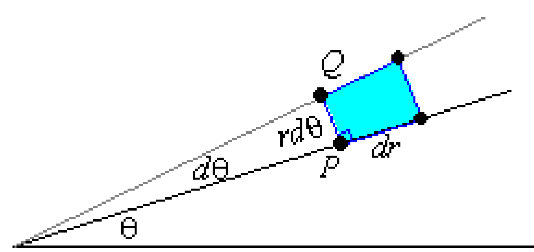
EXAMPLE 6 Find the Jacobian determinant and the area differential for the polar coordinate transformation. Illustrate using the image of a "grid" of rectangles in polar coordinates.

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the Jacobian determinant is

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= \cos(\theta)\, r\cos(\theta) - -r\sin(\theta)\sin(\theta) \\ &= r\left[\cos^2(\theta) + \sin^2(\theta)\right] \\ &= r\end{aligned}$$

Thus, the area differential is $dA = r dr d\theta$.
Geometrically, "rectangles" in polar coordinates are regions between circular arcs away from the origin and rays through the origin. If the distance changes from r to $r + dr$ for $r > 0$ and some small $dr > 0$, and if the polar angle changes from θ to $\theta + d\theta$ for some small angle $d\theta$, then the region covered is practically the same as a small rectangle with height dr and width ds, which is the distance from θ to $\theta + d\theta$ along a circle of radius r.





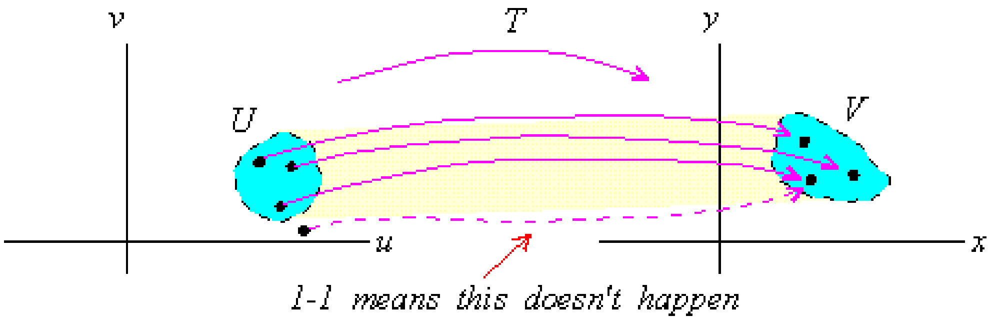
If an arc subtends an angle $d\theta$ of a circle of radius r, then the length of the arc is $ds = r d\theta$. Thus,

$$dA = dr\ ds = r dr d\theta$$

Check your Reading: Do "rectangles" in polar coordinates resemble rectangles if r is arbitrarily close to 0?

The Inverse Function Theorem

Recall taht if a coordinate transformation T maps an open region U in the uv-plane to an open region V in the xy-plane, then T is 1-1 if each point in V is the image of *only one* point in U.



Additionally, if every point in V is the image under $T(u,v)$ of at least one point in U, then $T(u,v)$ is said to map U *onto* V.

If $T(u,v)$ is a 1-1 mapping of a region U in the uv-plane **onto** a region V in the xy-plane, then we define the *inverse transformation* of T from V onto U by

$$T^{-1}(x,y) = (u,v) \quad only \quad if \quad (x,y) = T(u,v)$$

The Jacobian determinant can be used to determine if T has an inverse transformation T^{-1} on at least some small region about a given point.

> **Inverse Function Theorem:** Let $T(u,v)$ be a coordinate transformation on an open region S in the uv-plane and let (p,q) be a point in S. If
>
> $$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u,v)=(p,q)} \neq 0$$
>
> then there is an open region U containing (p,q) and an open region V containing $(x,y) = T(p,q)$ such that T^{-1} exists and maps V onto U.
>
> *image*

The proof of the inverse function theorem follows from the fact that the Jacobian matrix of $T^{-1}(x,y)$, when it exists, is given by the inverse of the Jacobian of T,

$$J^{-1}(x,y) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} \begin{bmatrix} y_v & -x_v \\ -y_u & x_u \end{bmatrix}$$

which features a Jacobian determinant with a negative power. Thus, J^{-1} exists only if the determinant of $J(u,v)$ is non-zero.

EXAMPLE 7 Where is $T(r,\theta) = \langle r\cos(\theta), r\sin(\theta)\rangle$ invertible?

Solution: The Jacobian determinant for polar coordinates is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

which is non-zero everywhere except the origin. Thus, at any point (r_0,θ_0) with $r_0 > 0$, there is an open region U in the $r\theta$-plane and an open region V containing $(x,y) = (r_0\cos(\theta_0), r_0\sin(\theta_0))$ such that $T^{-1}(x,y)$ exists and maps V onto U.

We will explore the result in example 7 more fully in the exercises. In particular, we will show that

$$T^{-1}(x,y) = \left\langle \sqrt{x^2+y^2}, 2\tan^{-1}\left(\frac{y}{x+\sqrt{x^2+y^2}}\right)\right\rangle$$

Clearly, T^{-1} is not defined on any open region containing $(0,0)$. Also, if $y=0$ and $x>0$, then

$$2\tan^{-1}\left(\frac{0}{x+\sqrt{x^2+0^2}}\right) = 2\tan^{-1}\left(\frac{0}{x+|x|}\right) = 0$$

But if $y=0$ and $x<0$, then

$$2\tan^{-1}\left(\frac{0}{x+\sqrt{x^2+0^2}}\right) = 2\tan^{-1}\left(\frac{0}{x+|x|}\right) = 2\tan^{-1}\left(\frac{0}{0}\right)$$

That is, a different representation of T^{-1} must be used on any region which intersects the negative real axis.

Exercises

Find the velocity vector in the uv-plane to the given curve. Then find Jacobian matrix and the tangent vector at the corresponding point to the image of the curve in the xy-plane.

1. $T(u,v) = \langle u+v, u-v\rangle$
   $u=t,\ v=t^2$ at $t=1$
2. $T(u,v) = \langle 2u+v, 3u-v\rangle$
   $u=t,\ v=t^2$ at $t=1$
3. $T(u,v) = \langle u^2v, uv^2\rangle$
   $u=t,\ v=3t$ at $t=2$
4. $T(u,v) = \langle u^2-v^2, 2uv\rangle$
   $u=\cos(t),\ v=\sin(t)$ at $t=0$
5. $T(u,v) = \langle u\sec(v), u\tan(v)\rangle$
   $u=t,\ v=\pi$ at $t=1$
6. $T(u,v) = \langle u\cosh(v), u\sinh(v)\rangle$
   $u=t,\ v=t^2$ at $t=1$

Find the Jacobian determinant and area differential of each of the following transformations.

7. $T(u,v) = \langle u+v, u-v \rangle$
8. $T(u,v) = \langle uv, u-v \rangle$
9. $T(u,v) = \langle u^2 - v^2, 2uv \rangle$
10. $T(u,v) = \langle u^3 - 3uv^2, 3u^2v - v^3 \rangle$
11. $T(u,v) = \langle ue^v, ue^{-v} \rangle$
12. $T(u,v) = \langle e^u \cos(v), e^u \sin(v) \rangle$
13. $T(u,v) = \langle 2u\cos(v), 3u\sin(v) \rangle$
14. $T(u,v) = \langle u^2\cos(v), u^2\sin(v) \rangle$
15. $T(u,v) = \langle e^u\cos(v), e^{-u}\sin(v) \rangle$
16. $T(u,v) = \langle e^u\cosh(v), e^{-u}\sinh(v) \rangle$
17. $T(u,v) = \langle \sin(u)\sinh(v), \cos(u)\cosh(v) \rangle$
18. $T(u,v) = \langle \sin(uv), \cos(uv) \rangle$

In each of the following, sketch several coordinate curves of the given coordinate system to form a grid of "rectangles" (i.e., make sure the u-curves are close enough to appear straight between the v-curves and vice-versa. Find the area differential and discuss its relationship to the "coordinate curve grid". (19 - 22 are linear transformations and have a constant Jacobian determinant)

19. $T(u,v) = \langle 2u, v \rangle$
20. $T(u,v) = \langle u+1, v \rangle$
21. $T(u,v) = \left\langle \frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}} \right\rangle$
22. $T(u,v) = \left\langle \frac{u-\sqrt{3}v}{2}, \frac{\sqrt{3}u+v}{2} \right\rangle$
23. *parabolic coordinates*
    $T(u,v) = \langle u^2 - v^2, 2uv \rangle$
22. *tangent coordinates*
    $T(u,v) = \left\langle \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right\rangle$
25. *elliptic coordinates*
    $T(u,v) = \langle \cosh(u)\cos(v), \sinh(u)\sin(v) \rangle$
24. *bipolar coordinates*
    $T(u,v) = \left\langle \frac{\sinh(v)}{\cosh(v)-\cos(u)}, \frac{\sin(u)}{\cosh(v)-\cos(u)} \right\rangle$

Some of the exercises below refer to the following formula for the inverse of the Jacobian:

$$J^{-1}(x,y) = \left(\frac{\partial(x,y)}{\partial(u,v)} \right)^{-1} \begin{bmatrix} y_v & -x_v \\ -y_u & x_u \end{bmatrix} \tag{4}$$

27. Find $T^{-1}(x,y)$ for the transformation

$$T(u,v) = \langle u+v, u-v \rangle$$

by letting $x = u+v$, $y = u-v$ and solving for u and v. Then find $J^{-1}(x,y)$ both (a) directly from $T^{-1}(x,y)$ and (b) from the formula (4).

28. Find $T^{-1}(x,y)$ for the transformation

$$T(u,v) = \langle u+4, u-v \rangle$$

Then find $J^{-1}(x,y)$ both (a) directly from $T^{-1}(x,y)$ and (b) from the formula (4).

29. At what points (u,v) does the coordinate transformation

$$T(u,v) = \langle e^u\cos(v), e^u\sin(v) \rangle$$

have an inverse? Can the same inverse be used over the entire uv-plane?

30. At what points (u,v) does the coordinate transformation

$$T(u,v) = \langle u\cosh(v), u\sinh(v)\rangle$$

have an inverse.

31. Show that if $T(u,v) = \langle au+bv, cu+dv\rangle$ where a,b,c,d are constants (i.e., $T(u,v)$ is a linear transformation), then $J(u,v)$ is the matrix of the linear transformation $T(u,v)$.

32. Show that if $T(u,v) = \langle au+bv, cu+dv\rangle$ where a,b,c,d are constants (i.e., $T(u,v)$ is a linear transformation), then

$$\frac{\partial(x,y)}{\partial(u,v)} = ad - bc$$

33. Show that if $f(u,v)$ is differentiable, then

$$\frac{\partial(f,f)}{\partial(u,v)} = 0$$

34. Show that if $f(u,v)$ and $g(u,v)$ are differentiable and if k is constant, then

$$\frac{\partial(kf,g)}{\partial(u,v)} = k\frac{\partial(f,g)}{\partial(u,v)}$$

35. Explain why if $x > 0$, then the inverse of the polar coordinate transformation is

$$T^{-1}(x,y) = \left\langle \sqrt{x^2+y^2},\ \ \tan^{-1}\left(\frac{y}{x}\right)\right\rangle$$

36. The Jacobian Matrix of $(r,\theta) = T^{-1}(x,y)$ is

$$K(x,y) = \begin{bmatrix} r_x & r_y \\ \theta_x & \theta_y \end{bmatrix}$$

Find $K(x,y)$ for $T^{-1}(x,y)$ in exercise 35, and then use polar coordinates to explain its relationship to

$$J^{-1}(r,\theta) = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

37. Show that if $x < 0$, then the inverse of the polar coordinate transformation is

$$T^{-1}(x,y) = \left\langle \sqrt{x^2+y^2},\ \ \pi + \tan^{-1}\left(\frac{y}{x}\right)\right\rangle$$

38. Use the following steps to show that if (x,y) is not at the origin or on the negative real axis, then

$$T^{-1}(x,y) = \left\langle \sqrt{x^2+y^2},\quad 2\tan^{-1}\left(\frac{y}{x+\sqrt{x^2+y^2}}\right)\right\rangle$$

is the inverse of the polar coordinate transformation.

a. Verify the identity

$$\tan(\phi) = \frac{\sin(2\phi)}{1 + \cos(2\phi)}$$

b. Let $\phi = \theta/2$ in a. Multiply numerator and denominator by r.
c. Simplify to an equation in x, y, and θ.

39. The coordinate transformation of rotation about the origin is given by

$$T(u, v) = \langle \cos(\theta) u + \sin(\theta) v, -\sin(\theta) v + \cos(\theta) u \rangle$$

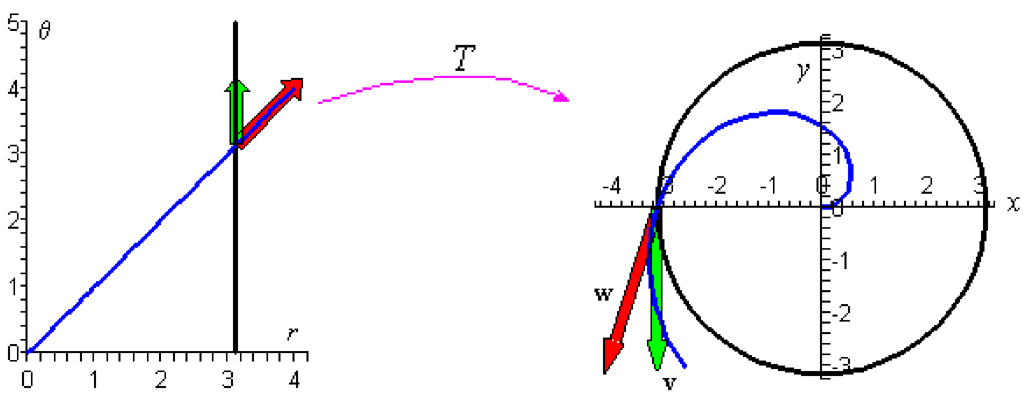
where θ is the angle of rotation. What is the Jacobian determinant and area differential for rotation through an angle θ? Explain the result geometrically.

40. The coordinate transformation of scaling horizontally by $a > 0$ and scaling vertically by $b > 0$ is given by

$$T(u, v) = \langle au, bv \rangle$$

What is its area differential? Explain the result geometrically.

41. A transformation $T(u, v)$ is said to be a *conformal transformation* if its Jacobian matrix preserves angles between tangent vectors. Consider that the vector $\langle 1, 0 \rangle$ is parallel to the line $r = \pi$ and that the vector $\langle 1, 1 \rangle$ is parallel to the line $r = \theta$. Also, notice that $r = \pi$ and $r = \theta$ intersect at $(r, \theta) = (\pi, \pi)$ at a $45°$ angle.



For $J(r, \theta)$ for polar coordinates, calculate

$$\mathbf{v} = J(\pi, \pi) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{w} = J(\pi, \pi) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Is the angle between $\mathbf{v}$ and $\mathbf{w}$ a $45°$ angle? Is the polar coordinate transformation conformal?

42. Find the Jacobian and repeat exercise 41 for the transformation

$$T(\rho, \theta) = \langle e^{\rho} \cos(\theta), e^{\rho} \sin(\theta) \rangle$$

43. Write to Learn: Write a short essay in which you calculate the area differential of the transformation $T(\rho,\theta) = \langle e^{\rho}\cos(\theta), e^{\rho}\sin(\theta)\rangle$ both computationally and geometrically.

44. Write to Learn: A coordinate transformation $T(u,v) = \langle f(u,v), g(u,)\rangle$ is said to be *area preserving* if the area of the image of any region R in the uv-plane is the same as the area of R. Write a short essay which uses the area differential to explain why a rotation through an angle θ is area preserving.

45. Proof of a Simplified Inverse Function Theorem: Suppose that the Jacobian determinant of $T(u,v) = \langle f(u,v), g(u,v)\rangle$ is non-zero at a point (p,q) and suppose that $\mathbf{r}(t) = \langle p+mt, q+nt\rangle$, t in $[-\varepsilon,\varepsilon]$, is a line segment in the uv-plane (m and n are numbers). Explain why if ε is sufficiently close to 0, then there is a 1-1 correspondence between the segment $\mathbf{r}(t)$ and its image $T(\mathbf{r}(t))$, t in $[-\varepsilon,\varepsilon]$. (Hint: first show that $x(t) = f(p+mt, q+nt)$ is monotone in t for t in $[-\varepsilon,\varepsilon]$).

46. Write to Learn: Let $T(u,v) = \langle x(u,v), y(u,v)\rangle$ be differentiable at $\mathbf{p} = (p,q)$ and assume that its Jacobian matrix is of the form

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

By letting $\mathbf{u} = \langle p+h, q\rangle$ in definition 5.1, (so that $\mathbf{u}-\mathbf{p} = [h \ \ 0]^t$ in matrix notation), show that

$$\lim_{\mathbf{u}\to\mathbf{p}} \frac{|T(\mathbf{u}) - T(\mathbf{p}) - J(\mathbf{p})(\mathbf{u}-\mathbf{p})|}{||\mathbf{u}-\mathbf{p}||} = 0$$

is transformed into

$$\lim_{h\to 0^+} \frac{||\langle x(p+h,q) - x(p,q), y(p+h,q) - y(p,q)\rangle - \langle ah, ch\rangle||}{h} = 0$$

Use this to show that $a = x_u$ and $c = y_u$. How would you find b and d? Explain your derivations and results in a short essay.