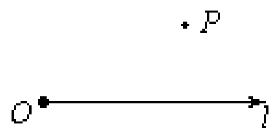
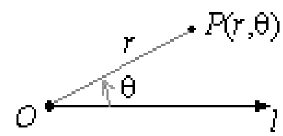
Polar Coordinates

Polar Coordinates

Although we explored the *polar coordinate transformation* as a type of coordinate transformation in the last section, polar coordinates occur frequently enough to warrant separate consideration. Let's begin by reviewing what it means to assign polar coordinates to a point the in the plane.

Suppose that l is a ray that begins at an origin O, and suppose that P is a point in the plane. Then P can be located with respect to l and o by the distance r from O to P and an angle θ formed by the segment OP and the ray l.



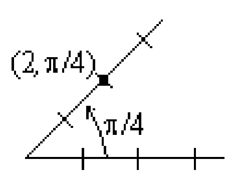


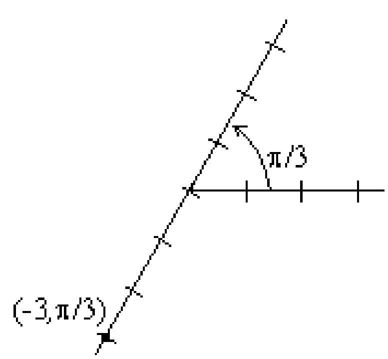
The order pair (r, θ) is the pair of *polar coordinates* of the point P.

The polar coordinates of a point are not unique. Since angles repeat every 2π radians, it follows that $(r, \theta) = (r, \theta + 2\pi)$. Moreover, a negative value of r implies a rotation of $180°$, so that we also have $(-r, \theta) = (r, \theta + \pi)$.

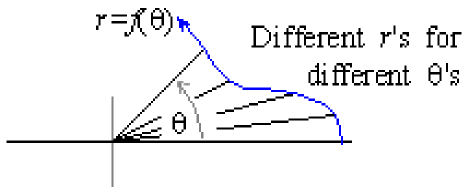
EXAMPLE 1 Locate the points $(2, \pi/4)$ and $(-3, \pi/3)$ using polar coordinates.

Solution: The point with polar coordinates $(2, \pi/4)$ must be a distance of 2 from the origin along the ray which is at an angle of $\pi/4$ from the x-axis. Likewise, the point with polar coordinates $(-3, \pi/3)$ is the same as the point $(3, \pi + \pi/3)$, as is shown below:

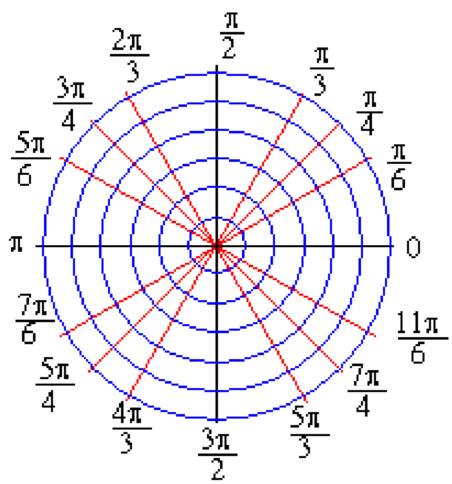




The graph of a polar function $f(\theta)$ is given by $r = f(\theta)$, which assigns a distance from the origin to each angle θ, thus forming a curve.



We can sketch the graph of a polar function by first calculating r at key angles and then plotting those points on a polar coordinate grid.



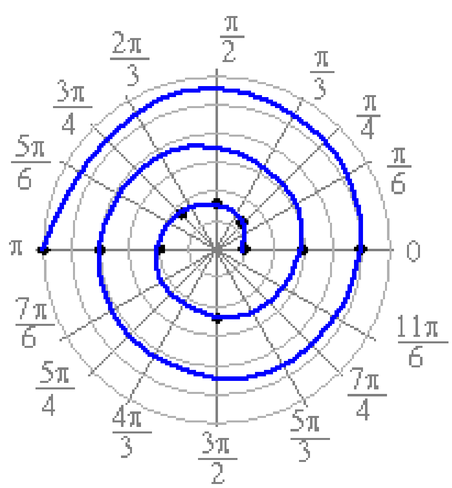
EXAMPLE 2 Sketch the graph of the function

$$r = \frac{1}{\pi}\theta + 1$$

Solution: First, let us choose some typical values of θ and compute r for those angles.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π	3π	4π	5π
r	1	1.25	1.5	1.75	2	2.5	3	4	5	6

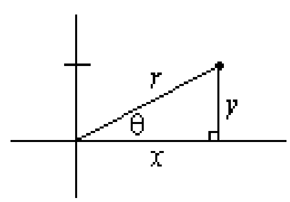
Now let's plot those points on the grid and connect them to obtain a sketch of the graph of the function:



Check your Reading: What type of curve is shown in example 2?

The Polar Coordinate Transformation

If we choose the ray l to be the positive x-axis, then a point P in the plane has both Cartesian coordinates (x, y) and polar coordinates (r, θ).



The definition of the sine and cosine functions imply that (x, y) is given in terms of (r, θ) by

$$x = r\cos(\theta), \qquad y = r\sin(\theta) \tag{1}$$

Solving for r and θ then yields the identities

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan(\theta) = \frac{y}{x} \tag{2}$$

EXAMPLE 3 Convert the point $(4, \pi/4)$ from polar coordinates into Cartesian coordinates, and then show that (2) converts it back into polar.

Solution: To do so, we let $r = 4$ and let $\theta = \pi/4$ in (1) to obtain

$$x = 4\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}, \qquad y = 4\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

To map back, we notice that

$$r^2 = x^2 + y^2 = 8 + 8 = 16, \quad r = 4$$

and $y/x = 1$ implies $\tan(\theta) = 1$, $\theta = \pi/4$.

If we substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into a curve $g(x, y) = k$, then the result

$$g(r\cos(\theta), r\sin(\theta)) = k$$

is called the *pullback* of the curve into polar coordinates. The identity $r^2 = x^2 + y^2$ is often used in pulling a curve back into polar coordinates.

For example, $x^2 + y^2 = R^2$ for a constant $R > 0$ has a pullback of

$$r^2 = R^2 \quad \Longrightarrow \quad r = R$$

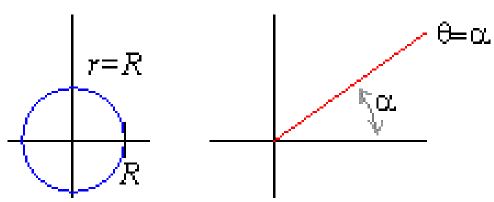
Similarly, lines of the form $y = mx$ become

$$r\sin(\theta) = mr\cos(\theta) \quad \Longrightarrow \quad \sin(\theta) = m\cos(\theta) \quad \Longrightarrow \quad \tan(\theta) = m$$

This matches example 4 in the last section, in which we saw that the coordinate curves for the *polar coordinate transformation*

$$T(r, \theta) = \langle r\cos(\theta), r\sin(\theta)\rangle$$

are circles centered at the origin and lines through the origin.



Also, it shows that whenever possible, we should solve for r to obtain a function of the form $r = f(\theta)$.

EXAMPLE 4 Convert the curve $x^2 + (y-1)^2 = 1$ into polar coordinates, and then solve for r, if possible.

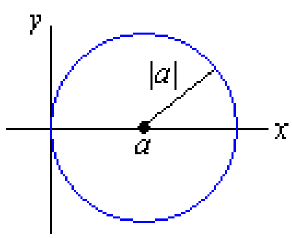
Solution: Expanding leads to $x^2 + y^2 - 2y + 1 = 1$, so that

$$r^2 - 2r\sin(\theta) + 1 = 1$$

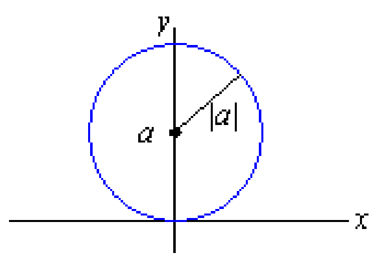
Solving for r then yields

$$r = 2\sin(\theta)$$

That is, $r = 2\sin(\theta)$ is a circle of radius 1 centered at $(0,1)$. In general, a curve of the form $r = 2a\cos(\theta)$ is a circle of radius $|a|$ centered at $(a,0)$ and a curve of the form $r = 2a\sin(\theta)$ is a circle of radius $|a|$ centered at $(0,a)$.



r=acos(θ)



r=asin(θ)

Check Your Reading: Where is the center of the circle $r = 2\cos(\theta)$.

Polar Coordinates in Vector Form

Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the graph of $r = f(\theta)$ is parametrized by

$$\mathbf{r}(\theta) = f(\theta)\ \langle\cos(\theta), \sin(\theta)\rangle, \quad \theta\ in\ [0, 2\pi] \tag{3}$$

Thus, tangent vectors, curvature, normals, arclength, etcetera, for polar curves $r = f(\theta)$ can be obtained by applying techniques and concepts in chapter 1 to $\mathbf{r}(\theta)$.

EXAMPLE 6 Find the pullback of $x = 1$ into polar coordinates. What is the velocity $\mathbf{v}$ for the pullback? What is significant about the result.

Solution: To do so, we let $x = r\cos(\theta)$, which corresponds to

$$r\cos(\theta) = 1 \quad \textit{or} \;\; r = \sec(\theta)$$

Thus, $\mathbf{r}(\theta) = \sec(\theta)\langle\cos\theta, \sin\theta\rangle = \langle 1, \tan(\theta)\rangle$ since $\sec\theta = 1/\cos\theta$ and $\tan(\theta) = \sin(\theta)\sec(\theta)$. Consequently,

$$\mathbf{v} = \frac{d}{d\theta}\mathbf{r} = \frac{d}{d\theta}\langle 1, \tan(\theta)\rangle = \left\langle 0, \sec^2(\theta)\right\rangle$$

Equivalently, $\mathbf{v} = \sec^2(\theta)\langle 0, 1\rangle = \sec^2(\theta)\,\mathbf{j}$. That is, the direction is constant (i.e., along the line $x = 1$), but the speed is not ($v = \sec^2(\theta)$).

If we let $\mathbf{e}_r = \langle\cos(\theta), \sin(\theta)\rangle$, then the parameterization of a polar curve is given by

$$\mathbf{r}(\theta) = r(\theta)\;\mathbf{e}_r$$

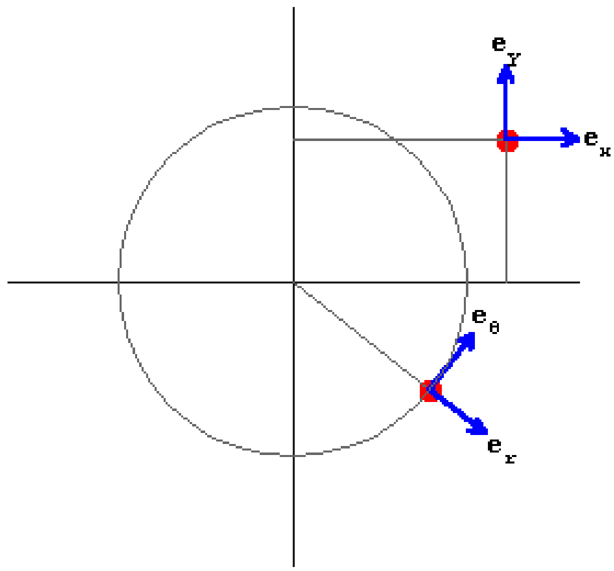
It follows that the velocity of $\mathbf{r}(\theta)$ is

$$\begin{aligned} \mathbf{v}(\theta) &= \left[\frac{d}{d\theta}r(\theta)\right]\langle\cos(\theta), \sin(\theta)\rangle + r(\theta)\frac{d}{d\theta}\langle\cos(\theta), \sin(\theta)\rangle \\ &= r'(\theta)\;\mathbf{e}_r + r(\theta)\langle -\sin(\theta), \cos(\theta)\rangle \qquad (4) \end{aligned}$$

If we let $\mathbf{e}_\theta = \langle -\sin(\theta), \cos(\theta)\rangle$,then (4) becomes

$$\mathbf{v}(\theta) = r'\;\mathbf{e}_r + r\;\mathbf{e}_\theta$$

The vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are *unit vectors* that satisfy $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$. Thus, for each value of θ, the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ form a *orthonormal basis* in polar coordinates, much like the vectors $\mathbf{i} = \mathbf{e}_x$ and $\mathbf{j} = \mathbf{e}_y$ do in Cartesian coordinates. However, as we move from point-to-point in the plane, the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ vectors change direction whereas the $\mathbf{e}_x$ and $\mathbf{e}_y$ vectors do not.

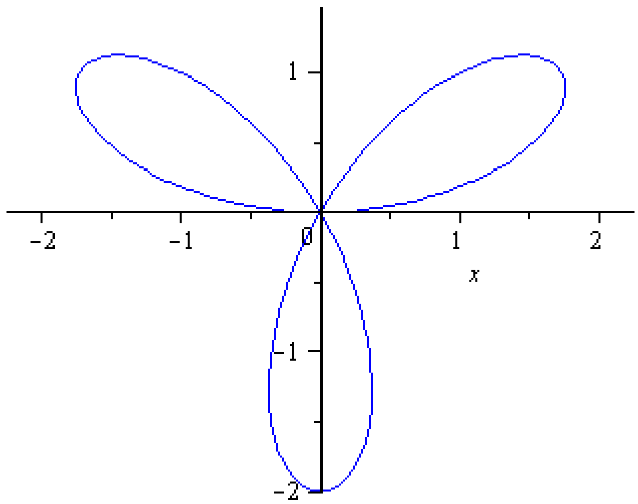


This reflects the fact that Cartesian coordinates are formed by 2 families of parallel lines, while polar coordinates are not.

EXAMPLE 7 Sketch the graph of $r = 2\sin(3\theta)$. What is the slope of the tangent line to the curve at $\theta = \pi/6$?

Solution: When $\theta = 0$ or $\theta = \pi/3$, then $r = 2\sin(0) = 2\sin(\pi) = 0$. Thus, the graph of $r = 2\sin(3\theta)$ forms a single loop for θ in $[0, \pi/3]$. It then follows that it forms another loop when θ is in

$[\pi/3, 2\pi/3]$ and yet another loop for θ in $[2\pi/3, \pi]$.



Substituting $r = 2\sin(3\theta)$ into (3) yields

$$\mathbf{r}(\theta) = 2\sin(3\theta)\,\mathbf{e}_r$$

which has a derivative of

$$\begin{aligned}\mathbf{v}(\theta) &= \frac{d}{d\theta}\left(2\sin(3\theta)\,\mathbf{e}_r\right)\\ &= 6\cos(3\theta)\,\mathbf{e}_r + 2\sin(3\theta)\,\mathbf{e}_\theta\end{aligned}$$

since $d\mathbf{e}_r/d\theta = \mathbf{e}_\theta$. Thus, at $\theta = \pi/6$ we have

$$\mathbf{v}\left(\frac{\pi}{6}\right) = 6\cos\left(\frac{\pi}{2}\right)\mathbf{e}_r + 2\sin\left(\frac{\pi}{2}\right)\mathbf{e}_\theta = 0 + 2\mathbf{e}_\theta$$

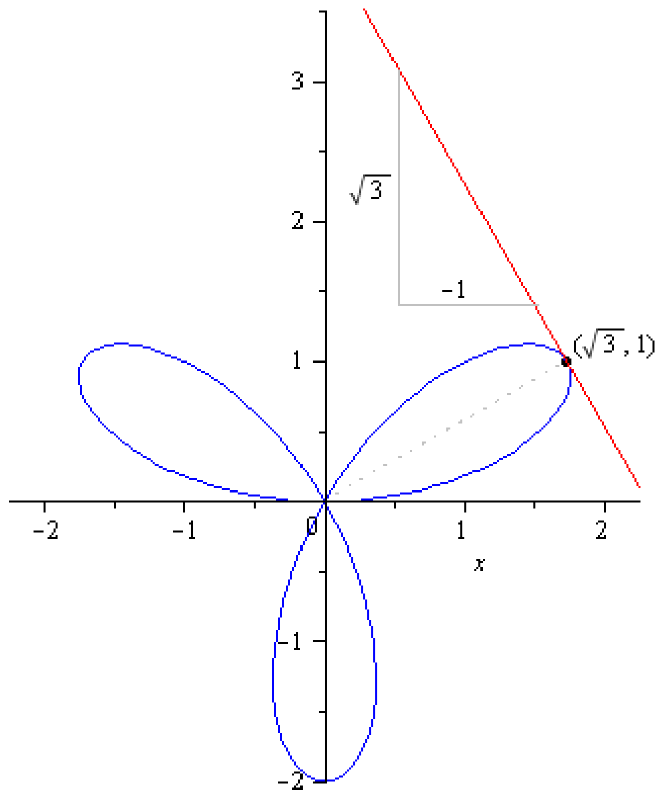
Evaluating $\mathbf{e}_\theta = \langle -\sin\theta, \cos\theta\rangle$ at $\theta = \pi/6$ leads to

$$\mathbf{e}_\theta = \left\langle \frac{-1}{2}, \frac{\sqrt{3}}{2}\right\rangle$$

so that

$$\mathbf{v}\left(\frac{\pi}{6}\right) = 2\left\langle \frac{-1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \left\langle -1, \sqrt{3}\right\rangle$$

Thus, the tangent line has a slope of $m = -\sqrt{3}$.



In example 7, we could have used the expanded form

$$\mathbf{r}(\theta) = 2\sin(3\theta)\ \langle\cos\theta, \sin\theta\rangle$$

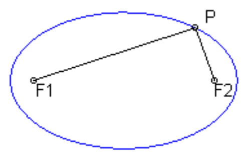
when determining $\mathbf{v}$. The basis notation $\mathbf{e}_r, \mathbf{e}_\theta$ is simply a more compact method for doing same thing.

Check Your Reading: How did we obtain $m = -\sqrt{3}$ from the velocity vector in example 7?

Ellipses in Polar Coordinates

Let's suppose that 2 "nails" are driven into a board at points F_1 and F_2, and suppose that the ends of a string of length $2a$ is attached to the board at points F_1 and F_2. If the string is pulled tight around a pencil's tip, then the points P

traced out by the pencil as it moves within the string is called an *ellipse.*

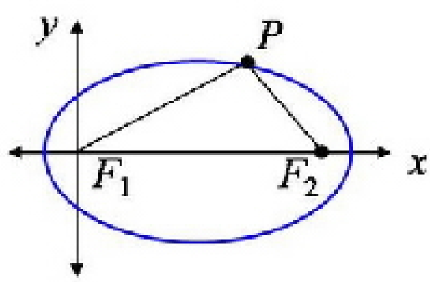


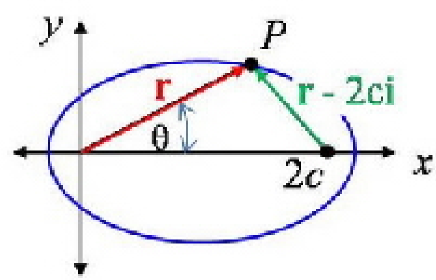
That is, an ellipse is the locus of all points P such that

$$|PF_1| + |PF_2| = 2a$$

where $|PF_1|$ and $|PF_2|$ denote distances from P to F_1 and F_2, respectively. The points F_1 and F_2 are called the *foci* of the ellipse, and the distance a is called the *semi-major axis.*

Let's use this definition of an ellipse to derive its representation in polar coordinates. To begin with, let's assume that F_1 is at the origin and that F_2 is on the positive real axis at the point $(2c, 0)$ (i.e., $2c$ is the distance from F_1 to F_2). Then $\mathbf{r}$ is the polar vector to the point P, and $\mathbf{r} - 2c\mathbf{i}$ is the vector from F_2 to P.





The ellipse definition implies that

$$\|\mathbf{r}\| + \|\mathbf{r} - 2c\mathbf{i}\| = 2a$$

Thus, $r = \|\mathbf{r}\|$ implies that $r - 2a = \|\mathbf{r} - 2c\mathbf{i}\|$, so that

$$(r - 2a)^2 = \|\mathbf{r} - 2c\mathbf{i}\|^2$$

However, $\mathbf{r} = \langle r\cos(\theta), r\sin(\theta)\rangle$ implies that

$$\begin{aligned}
(r - 2a)^2 &= (r\cos\theta - 2c)^2 + (r\sin\theta)^2 \\
(r - 2a)^2 &= r^2\cos^2(\theta) - 2rc\cos(\theta) + 4c^2 + r^2\sin^2(\theta) \\
r^2 - 4ar + 4a^2 &= r^2 - 4rc\cos(\theta) + 4c^2
\end{aligned}$$

Simplifying and solving for r yields

$$\begin{aligned} -4ar + 4a^2 &= -4rc\cos(\theta) + 4c^2 \\ -ar + rc\cos(\theta) &= -a^2 + c^2 \\ r(-a + c\cos(\theta)) &= -a^2 + c^2 \\ r &= \frac{-a^2 + c^2}{-a + c\cos(\theta)} \end{aligned}$$

If we let $b^2 = a^2 - c^2$ denote the square of the *semi-minor* axis, then

$$r = \frac{b^2}{a - \cos(\theta)}$$

Finally, let us divide by a to obtain

$$r = \frac{b^2/a}{1 - c/a\cos(\theta)}$$

Usually, we let $\varepsilon = c/a$ and let $p = b^2/a$, where ε is called the *eccentricity* of the ellipse and p is called the *parameter*.

Thus, an ellipse in polar coordinates with one focus at the origin and the other on the non-negative x-axis is given by

$$r = \frac{p}{1 - \varepsilon\cos(\theta)}$$

Also, it follows that $0 \leq \varepsilon < 1$ and $p > 0$.

EXAMPLE 7 Find the eccentricity, parameter, and equation of the ellipse with foci at $(0, 0)$ and $(8, 0)$ and a semi-major axis of 5. What is the cartesian equation of the ellipse?

Solution: Since $2c$ is the distance between the foci, we have $a = 5$ and $c = 4$. The semi-minor axis satisfies

$$b^2 = 25 - 16 = 9 \quad or \quad b = 3$$

Thus, the parameter is $p = 9/5$ and the eccentricity is $\varepsilon = 4/5$, so that

$$r = \frac{9/5}{1 - 4/5\cos(\theta)} = \frac{9}{5 - 4\cos(\theta)}$$

To convert to x and y, we first multiply to get

$$5r - 4r\cos(\theta) = 9$$

Thus, $5r = 4r\cos(\theta) + 9$, so that

$$\begin{aligned} 25r^2 &= (4r\cos(\theta) + 9)^2 \\ 25\left(x^2 + y^2\right) &= (4x + 9)^2 \end{aligned}$$

Since $(4x+9)^2 = 16x^2 + 72x + 81$, we have

$$25x^2 + 25y^2 = 16x^2 + 72x + 81$$

which yields $9x^2 - 72x + 25y^2 = 81$.

Moreover, $b^2 = a^2 - c^2$ implies that

$$p = a\left(\frac{b^2}{a^2}\right) = a\left(\frac{a^2 - c^2}{a^2}\right) = a\left(1 - \frac{c^2}{a^2}\right),$$

which in turn implies that $p = a\left(1 - \varepsilon^2\right)$.Also, had F_2 been placed on the positive y-axis, the negative x-axis, or the negative y-axis, the polar equation of the ellipse would be modified by replacing $\cos(\theta)$ with $\sin(\theta)$, $-\cos(\theta)$, or $-\sin(\theta)$, respectively.

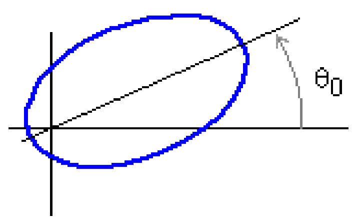
Polar Ellipse with F_2 on the

positive x-axis	$r = \frac{p}{1-\varepsilon\cos(\theta)}$	positive y-axis	$r = \frac{p}{1-\varepsilon\sin(\theta)}$
negative x-axis	$r = \frac{p}{1+\varepsilon\cos(\theta)}$	negative y-axis	$r = \frac{p}{1+\varepsilon\sin(\theta)}$

Indeed, the most general form of an conic with parameter p and eccentricity $\varepsilon > 0$ is

$$r = \frac{p}{1 - \varepsilon\cos(\theta - \theta_0)} \tag{5}$$

where it can be shown that $a = \varepsilon\cos(\theta_0)$ and $b = \varepsilon\sin(\theta_0)$. It follows that the conic is symmetric about the line at angle θ_0 to the x-axis.



EXAMPLE 8 Find the center, semi-major axis, semi-minor axis and foci of the ellipse

$$r = \frac{16}{5 + 3\sin(\theta)}$$

Solution: Let's divide the numerator and denominator by 5 to obtain

$$r = \frac{16/5}{1 + 3/5\sin(\theta)}$$

Since $p = 16/5$ and $\varepsilon = 3/5$, the formula $p = a\left(1 - \varepsilon^2\right)$ implies that

$$a = \frac{16/5}{1 - (3/5)^2} = 5$$

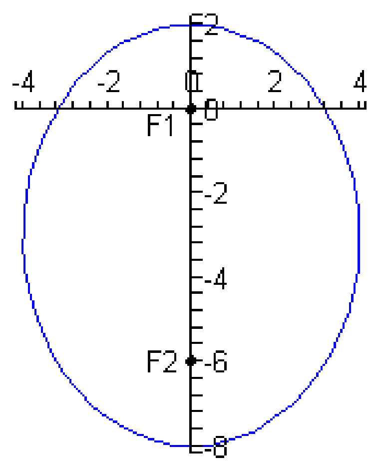
As a result, $b^2 = ap$ implies that

$$b^2 = 5 \cdot \left(\frac{16}{5}\right) = 16, \qquad b = 4$$

Finally, the focus F_1 is at the origin, and the table above implies that F_2 is on the negative y-axis. Since

$$2\varepsilon a = 2 \cdot \left(\frac{3}{5}\right) \cdot 5 = 6$$

F_2 is at the point $(0, -6)$.



Exercises

1. *Transform the point into xy-coordinates.*

a.	$(2, \pi)$	c.	$(3, \pi/6)$
b.	$(1, 9\pi)$	d.	$(-2, 2\pi/3)$

2. *Transform the point into polar coordinates.*

a.	$(2, 2)$	c.	$\left(-\sqrt{5}, -\sqrt{5}\right)$
b.	$\left(-\sqrt{3}, 1\right)$	d.	$(-15, 0)$

Use a polar grid to sketch the graph of the following polar functions. Then find the velocity vector $\mathbf{v}(\theta)$ *at the given* θ*. Compare your sketch to the plot produced by the "polar plotting tool" in the "Tools" section.*

3.	$r = 5, \quad \theta = \frac{\pi}{4}$	4.	$r = \frac{\theta^2}{\pi^2}, \quad \theta = \frac{\pi}{6}$
5.	$r = \frac{6}{\pi}\theta, \quad \theta = \frac{\pi}{6}$	6.	$r = \frac{\theta^2}{\pi^2}, \quad \theta = \frac{\pi}{6}$
7.	$r = 1 + \cos(\theta), \quad \theta = \frac{\pi}{2}$	8.	$r = \cos(2\theta), \quad \theta = \frac{\pi}{4}$
9.	$r = 2 + \cos(\theta), \quad \theta = \frac{2\pi}{3}$	10.	$r = \cos(3\theta), \quad \theta = \frac{7\pi}{4}$

Find the pullback of the following curves and then solve for r *to obtain the curve's functional form.*

11.	$x^2 + y^2 = 16$	12.	$x^2 + 3xy = y^2$
13.	$x = 1$	14.	$y = 1$
15.	$y = 3x + 2$	16.	$x^2 - y^2 = 1$
17.	$y = x^2$	18.	$y = x^3$
19.	$x = \frac{y^2}{4} - 1$	20.	$y = x^2 - \frac{1}{4}$
21.	$x^2 + y^2 = 2y + 3$	22.	$x^2 + y^2 = 4x + 2y$
23.	$(x-1)^2 + y^2 = 4$	24.	$x^2 + y^2 = (x+y)^2$
25.	$xy = 2$	26.	$3x^2 + 4y^2 - 4x = 4$

Identify the eccentricity and the parameter of each ellipse. Also, calculate the semi-major and semi-minor axes, the location of the foci, and the center. Finally, sketch the ellipse and convert it into xy-coordinates.

27.	$r = \frac{12}{1 - 3\cos(\theta)}$	28.	$r = \frac{12}{3 - \cos(\theta)}$
29.	$r = \frac{2}{1 - \sin(\theta)}$	30.	$r = \frac{2}{3 - 2\sin(\theta)}$
31.	$r = \frac{4}{2 + \sin(\theta)}$	32.	$r = \frac{1}{1 + \cos(\theta)}$
33.	foci at $(0,0)$ and $(6,0)$ semi-major axis of 5	34.	foci at $(0,0)$ and $(0,10)$ semi-major axis of 13

35. Show that a polar function of the form

$$r = \frac{p}{1 - \sin(\theta)}$$

is a parabola. Where is its vertex?

36. Which curve in the xy-plane is represented by the polar equation $r = \sec(\theta)\tan(\theta)$?

37. Show that

$$\frac{d}{d\theta}\mathbf{e}_\theta = -\mathbf{e}_r$$

38. Show that if r and θ are functions of t and $\mathbf{r}(t) = \langle r\cos(\theta), r\sin(\theta)\rangle$, then

$$\frac{d}{dt}\mathbf{e}_r = \mathbf{e}_\theta \frac{d\theta}{dt} \quad \textit{and} \quad \frac{d}{dt}\mathbf{e}_\theta = -\mathbf{e}_r \frac{d\theta}{dt}$$

39. Show that if r and θ are functions of t and $\mathbf{r}(t) = \langle r\cos(\theta), r\sin(\theta)\rangle$, then

$$\begin{aligned}\mathbf{v} &= \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta \\ \mathbf{a} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta\end{aligned}$$

(Hint: See exercise 38).

40. Show that $y = mx + b$ is pulled back into polar coordinates to the function

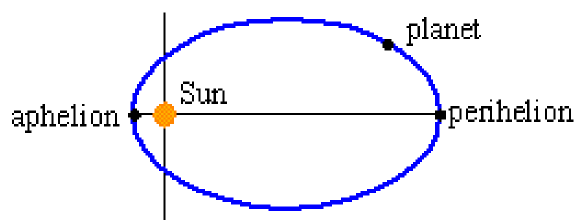
$$r = \frac{b}{\sin(\theta) - m\cos(\theta)}$$

At what value of θ does r not exist? What is significant about this value of θ?

41. The orbit of the earth about the sun is an ellipse with the sun's center as one focus. The parameter of the earth's orbit is $185,740,000$ miles and the eccentricity is 0.017. What is the semi-major axis, the semi-minor axis, and the foci of the earth's orbit.

42. A satellite has an orbit with a parameter of $4,335$ miles and an eccentricity of 0.067. If we assume the earth is a sphere with a radius of $3,963$ miles, what is the closest the satellite comes to the earth?

43. The positions of a planet closest to and farthest from the sun are called its *perihelion* and *aphelion*, respectively.



Show that if $0 < \varepsilon < 1$ and if the planet's orbit is given by

$$r = \frac{p}{1 - \varepsilon\cos(\theta)}$$

then the perihelion distance is $a(1-\varepsilon)$ and the aphelion distance is $a(1+\varepsilon)$.

44. Pluto is at a distance of 4.43×10^9 km when it is closest to the Sun and is at a distance 7.37×10^9 km when it is farthest from the sun. Use the result in exercise 43 to determine the eccentricity of the orbit.

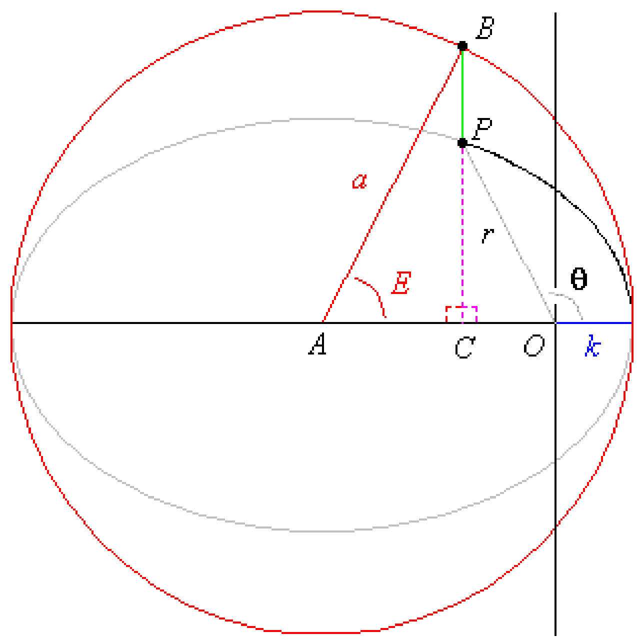
45. Write to Learn: Write a short essay in which you derive and explain the following identities:

$$\begin{aligned} a &= \frac{r(0) + r(\pi)}{2}, \quad \varepsilon a = \frac{r(0) - r(\pi)}{2}, \\ b^2 &= r(0)\,r(\pi), \quad \varepsilon = \frac{r(0) - r(\pi)}{r(0) + r(\pi)}\end{aligned}$$

46. The Eccentric Anomaly: The equation in polar coordinates of an ellipse centered on the negative x-axis is

$$r = \frac{p}{1 + \varepsilon \cos(\theta)}$$

Let $a = p/\left(1 - \varepsilon^2\right)$ and let $k = p/\left(1 + \varepsilon\right)$. The *auxiliary circle* of the ellipse is the circle centered at $(k - a, 0)$ with radius a, and the *eccentric anomaly* of an ellipse is the angle E formed with by the radius a of the circle that terminates on the vertical line through the point $P(r, \theta)$ (see figure below):



polarex3

animation created by Clayton Clark, Click to see ani, Click to return

The eccentric anomaly is important in celestial mechanics, where we often reparameterize r as a function of the angle E.

1. (a) Show that $r\cos(\theta) = \alpha\cos(E) - a\varepsilon$ (hint: in the figure above, $r\cos(\pi - \theta) > 0$ and $a\cos(E) > 0$).

   (b) Use (a) and the fact that $r + \varepsilon r \cos(\theta) = p$ to show that

$$r = p + a\varepsilon^2 - a\varepsilon\cos(E)$$

(c) Show that $p = a\left(1 - \varepsilon^2\right)$, so that (b) implies that

$$r = a - a\varepsilon\cos(E)$$

(d) Use the definition of k to show that $a\varepsilon = a - k$ and that (c) can be written in the form

$$r = a - (a - k)\cos(E) \tag{6}$$

(e) Explain why the segment BP in the figure above has a length of

$$|BP| = (a - k)\,|\cos(E)|$$

How might (6) and the length of BP be used to suggest a different method for constructing an ellipse?