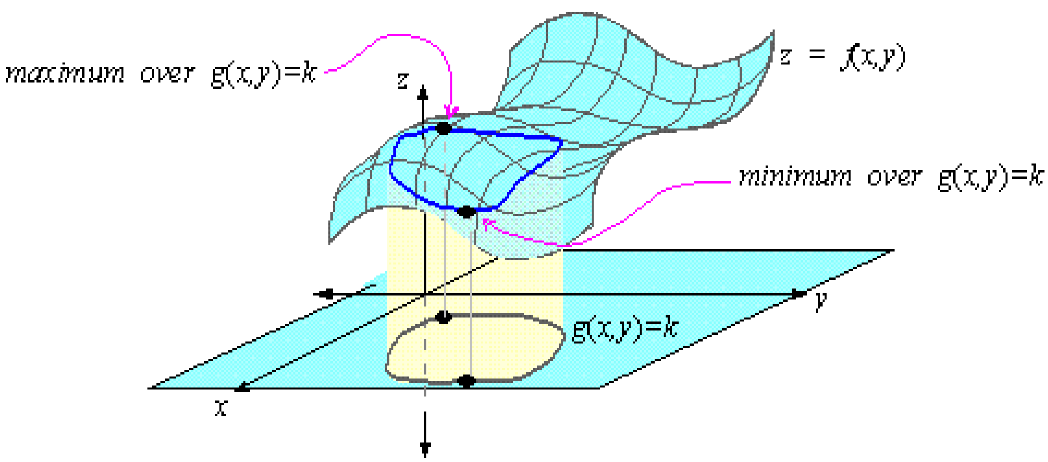
Lagrange Multipliers

Optimization with Constraints

In many applications, we must find the extrema of a function $f(x,y)$ subject to a *constraint* $g(x,y)=k$. Such problems are called *constrained optimization* problems.

For example, suppose that the constraint $g(x,y)=k$ is a smooth closed curve parameterized by $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ on $[a,b]$, and suppose that $f(x,y)$ is differentiable at each point on the constraint. Then finding the extrema of $f(x,y)$ subject to $g(x,y)=k$ is equivalent to finding the absolute extrema of the function $z(t)=f(x(t),y(t))$ for t in $[a,b]$.



In a first calculus course, we learn that the extrema of $z(t)$ over $[a,b]$ must exist and occur either at the critical points or the endpoints of $[a,b]$. Since the curve is closed, we only need consider the critical points of $z(t)$ in $[a,b]$, which are solutions to

$$\frac{dz}{dt}=\nabla f\cdot\mathbf{v}=0$$

where $\mathbf{v}$ is the velocity of $\mathbf{r}(t)$. That is, the critical points of $z(t)$ occur when $\nabla f\perp\mathbf{v}$. Since also $\nabla g\perp\mathbf{v}$, it follows that the extrema of $f(x,y)$ subject to $g(x,y)=k$ occur when ∇f is parallel to ∇g.

If ∇f is parallel to ∇g, then there is a number λ for which

$$\nabla f=\lambda\nabla g$$

Thus, the extrema of $f(x,y)$ subject to $g(x,y)=k$ must occur at the points which are the solution to the system of equations

$$\langle f_x,f_y\rangle=\lambda\langle g_x,g_y\rangle,\qquad g(x,y)=k \tag{1}$$

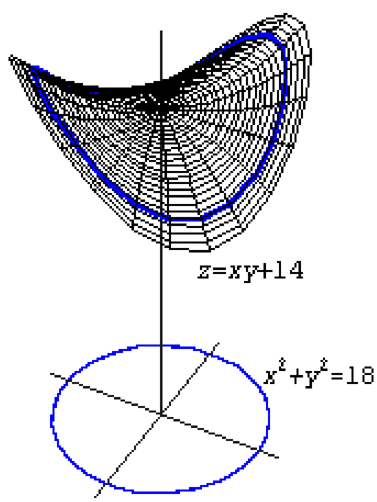
We call (1) a *Lagrange multiplier problem* and we call λ a *Lagrange Multiplier.*

A good approach to solving a Lagrange multiplier problem is to *first eliminate the Lagrange multiplier* λ using the two equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$. Then solve for x and y by combining the result with the constraint $g(x,y) = k$, thus producing the critical points. Finally, since the constraint $g(x,y) = k$ is a closed curve, the extrema of $f(x,y)$ over $g(x,y) = k$ are the largest and smallest values of $f(x,y)$ evaluated at the critical points.

EXAMPLE 1 Find the extrema of $f(x,y) = xy + 14$ subject to

$$x^2 + y^2 = 18$$

Solution: That is, we want to find the highest and lowest points on the surface $z = xy + 14$ over the unit circle:



If we let $g(x,y) = x^2 + y^2$, then the constraint is $g(x,y) = 18$. The gradients of f and g are respectively

$$\nabla f = \langle y, x\rangle \qquad \text{and} \qquad \nabla g = \langle 2x, 2y\rangle$$

As a result, $\nabla f = \lambda \nabla g$ implies that

$$y = \lambda 2x \quad \textit{and} \quad x = \lambda 2y$$

. Clearly, $x = 0$ only if $y = 0$, but $(0,0)$ is not on the unit circle. Thus, $x \neq 0$ and $y \neq 0$, so that solving for λ yields

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y} \qquad \Longrightarrow \qquad \frac{y}{2x} = \frac{x}{2y}$$

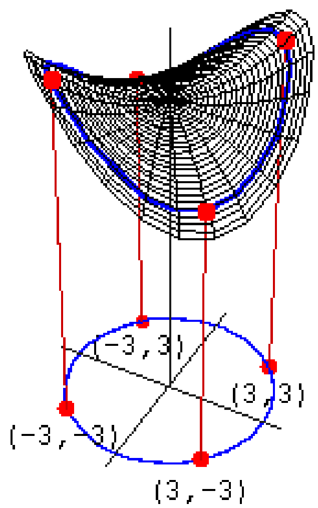
Cross-multiplying then yields $2y^2 = 2x^2$, which is the same as $y^2 = x^2$. Thus, the constraint $x^2 + y^2 = 18$ becomes

$$x^2 + x^2 = 18, \qquad x^2 = 9, \qquad x = \pm 3$$

Moreover, $y^2 = x^2$ implies that either $y = x$ or $y = -x$, so that the solutions to (2) are

$$(3,3)\,,\ (-3,3)\,,\ (3,-3)\,,\ (-3,-3)$$

However, $f(3,3) = f(-3,-3) = 23$, while $f(-3,3) = f(3,-3) = 5$. Thus, the maxima of $f(x,y) = xy + 4$ over $x^2 + y^2 = 18$ occur at $(3,3)$ and $(-3,-3)$, while the minima of $f(x,y) = xy + 4$ occur at $(-3,3)$ and $(3,-3)$.



More generally, finding the extrema of a differentiable function $f(x,y)$ subject to a constraint $g(x,y) = k$ is defined in terms of a *Lagrangian*, which is a function of *three variables* x, y, and λ of the form

$$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - k)$$

This is because the critical points of $L(x,y,\lambda)$ occur when

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0$$

and $L_x = f_x - \lambda g_x$, $L_y = f_y - \lambda g_y$, and $L_\lambda = g(x,y) - k$. That is, the critical points of $L(x,y,\lambda)$ are solutions of the system of equations

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \quad g(x,y) = k \tag{2}$$

which is the same as the equations in (1).

blue*EXAMPLE 2* blackShow that the Lagrangian for the problem of finding the extrema of $f(x,y) = xy + 14$ subject to

$$x^2 + y^2 = 18$$

reduces to the solution in example 1.

Solution: The Lagrangian for example 1 is

$$L(x,y,\lambda) = xy + 14 - \lambda\left(x^2 + y^2 - 18\right)$$

and correspondingly, $L_x = y - \lambda(2x)$, $L_y = y - \lambda(2y)$, and

$$L_\lambda = -\left(x^2 + y^2 - 18\right)$$

The critical points of L satisfy $L_x = 0$, $L_y = 0$, and $L_\lambda = 0$, which results in

$$y = \lambda 2x \quad and \quad x = \lambda 2y$$

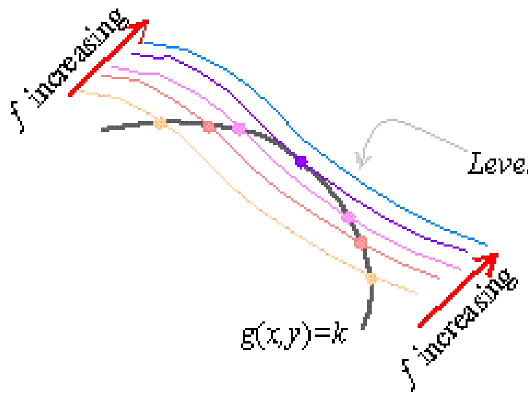
along with $x^2 + y^2 = 18$. The remainder of the solution is the same as in example 1.

Check your Reading: Can you identify the maxima and minima on the graph shown above.

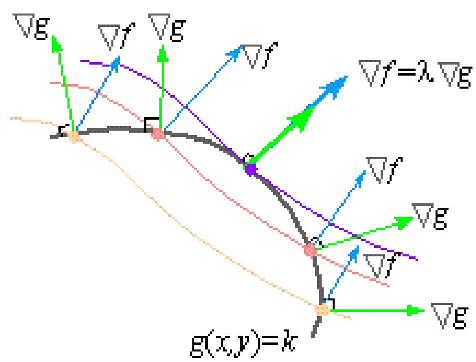
Lagrange Multipliers and Level Curves

Let's view the Lagrange Multiplier method in a different way, one which only requires that $g(x,y) = k$ have a *smooth* parameterization $\mathbf{r}(t)$ with t in a closed interval $[a,b]$. Such constraints are said to be *smooth and compact.*

If $f(x,y)$ is differentiable and to be optimized subject to a smooth compact constraint $g(x,y) = k$, then as the levels of $f(x,y)$ increase, short sections of level curves of $f(x, y)$ form secant curves to $g(x, y) = k$



It follows that the highest level curve of $f(x, y)$ intersecting a section of $g(x, y) = k$ must be tangent to the curve $g(x, y) = k$, which is possible only if their gradients ∇f and ∇g are parallel.



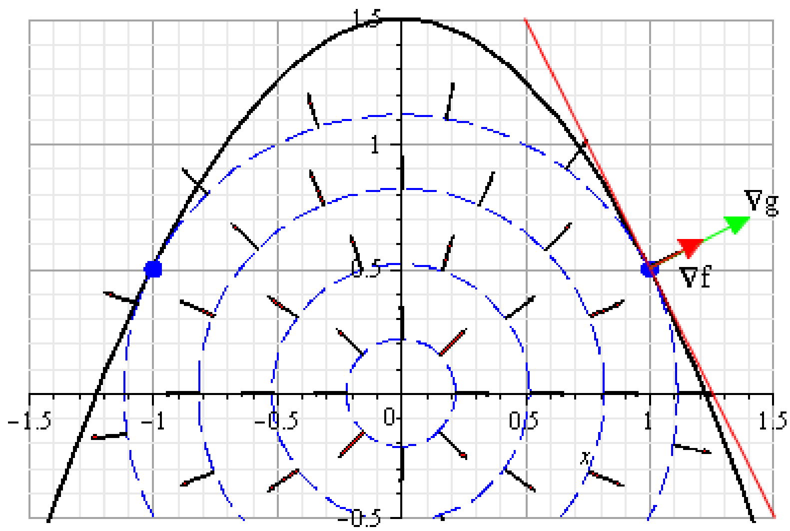
Consequently, if $g(x, y) = k$ is smooth and compact, then finding the extrema of $f(x, y)$ subject to $g(x, y) = k$ is equivalent to finding the critical points onf the function $L(x, y, \lambda)$, and then evaluating $f(x, y)$ at those critical points (and if applicable the boundary points of $g(x, y) = k$).

EXAMPLE 3 Find the point(s) on the curve $y = 1.5 - x^2$ closest to the origin both visually and via the Lagrange Multiplier method.

Solution: If we let $f(x, y)$ be the square of the distance from a point (x, y) to the origin $(0, 0)$, then our constrained optimization problem is to

$$\begin{aligned} \text{Minimize } f(x,y) &= x^2+y^2 \\ \text{Subject to } x^2+y &= 1.5 \end{aligned}$$

We will thus let $g(x,y) = x^2 + y$. Graphically, we can find the point on $y = 1.5 - x^2$ closest to the origin by drawing concentric circles centered at the origin with ever greater radii until they interesect the curve. The first intersection of a circle with the curve will correspond to a circle tangent to the curve – i.e., a point where ∇f is parallel to ∇g.



Points with $|x| > 1.5$ are more distant than any of the points for $|x| \leq 1.5$, so the point $(0.5, 1)$ and $(-0.5, 1)$ are the closest to the origin. Alternatively, the Lagrangian for this problem is

$$L(x,y,\lambda) = x^2 + y^2 - \lambda\left(x^2 + y - 1.5\right)$$

Since $L_x = 2x - \lambda(2x)$, $L_y = 2y - \lambda(1)$, and $L_\lambda = x^2 + y - 1.5$, the critical points of L occur when

$$2x = \lambda 2x, \qquad 2y = \lambda, \qquad x^2 + y = 1.5$$

To eliminate λ, we substitute the second equation $\lambda = 2y$ into the first equation to obtain

$$2x = (2y)\,2x, \qquad x = 2xy$$

If $x = 0$, then $y = 1.5$. Thus, $(0, 1.5)$ is the critical point corresponding to $\lambda = 3$. If $x \neq 0$, then $1 = 2y$ so that $y = 0.5$ and

$$x^2 + 0.5 = 1.5, \qquad x = \pm 1$$

Thus, the critical points are $(0,-1)$, $(1,0.5)$ and $(-1,0.5)$. However,

$$f(1,0.5) = f(-1,0.5) = 1.25, \qquad f(0,1.5) = 2.25$$

Since we need only consider points with $|x| \leq 1.5$, the points on $y = 1.5 - x^2$ which are closest to the origin are

$$(1,0.5) \quad \text{and} \quad (-1,0.5)$$

Similarly, an equation of the form $g(x,y,z) = k$ defines a *level surface* in 3-dimensions, and finding the extrema of a function $f(x,y,z)$ subject to a constraint $g(x,y,z) = k$ is equivalent to finding a level surface of $f(x,y,z)$ that is tangent to a constraint surface $g(x,y,z) = k$. It follows that ∇f is parallel to ∇g at this point, so that if $g(x,y,z) = k$ is a *closed, bounded* surface (i.e,. a surface with finite extent that contains its boundary points), then solving

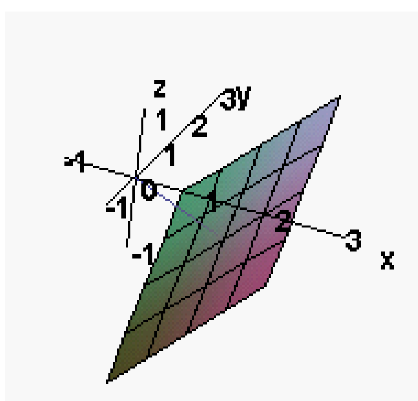
$$\begin{aligned} \textit{Optimize } w &= f(x,y,z) \\ \textit{Subject to } g(x,y,z) &= k \end{aligned}$$

is equivalent to finding the critical points of the Lagrangian in 4 variables given by

$$L(x,y,z,\lambda) = f(x,y,z) - \lambda(g(x,y,z) - k)$$

and then evaluating $f(x,y,z)$ at those critical points (and if applicable the boundary points of $g(x,y,z) = k$). Let's revisit a problem from the previous section to see this idea at work.

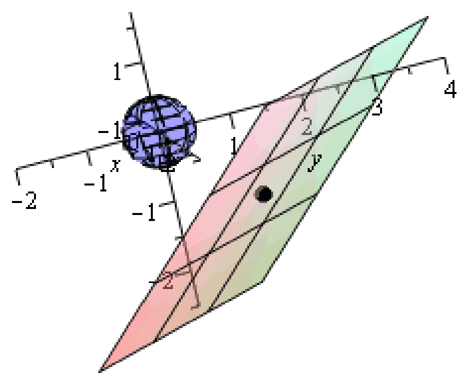
blue*EXAMPLE 4* blackFind the point(s) on the plane $x+y-z=3$ that are closest to the origin.



Solution: To begin with, we let f denote the *square* of the distance from a point (x, y, z) to the origin. Consequently, $f = x^2 + y^2 + z^2$. Thus, we want to

$$\begin{aligned} \text{minimize } f(x, y, z) &= x^2 + y^2 + z^2 \\ \text{subject to: } & \quad x + y - z = 3 \end{aligned}$$

Graphically, we can locate the closest point by drawing concentric spheres expanding until they intersect the plane, thus resulting in the "closest point" on a sphere that is tangent to the surface.



Alternatively, we can define the Lagrangian to be

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y - z - 3)$$

Since $L_x = 2x - \lambda$, $L_y = 2y - \lambda$, $L_z = 2z + \lambda$, and $L_\lambda = -(x + y - z - 3)$, the critical points are solutions to

$$2x = \lambda,\ 2y = \lambda,\ -2z = \lambda,\ \textit{and}\ x + y - z = 3 \tag{3}$$

To eliminate λ, we notice that the first two equations imply that $y = z$, while the first and third imply that $z = -x$. Substituting into the constraint (the last equation in (3)) leads to

$$x + x - (-x) = 3, \quad 3x = 3, \ x = 1$$

Thus, $y = 1$ and $z = -1$, so the critical point is $(1, 1, -1)$. Points of the plane not shown in the figure above are further from the origin than the points that are shown in the figure, and the section of the plane shown in the figure is a closed subset of the plane whose boundary points (i.e., the edges) are more than $\sqrt{3}$ away from the origin. Since the set is closed, a minimum of $f(x, y, z)$ must occur, yet it cannot occur on the boundary. Instead, it must occur at the critical point $(1, 1, -1)$ which is a distance of $\sqrt{3}$ from the origin.

Check your Reading: Can $\lambda = 0$ in example 4?

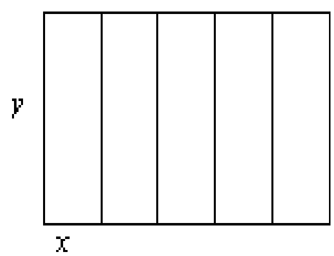
Applications

Many of the optimization word problems in a first calculus course are, in fact, constrained optimization problems of the form

$$\begin{array}{lcl} \textit{Optimize} & : & f(x,y) \\ \textit{Subject to} & : & g(x,y) = k \end{array}$$

In such problems, the Lagrange multiplier method produces a family of "extrema" of $f(x,y)$ parameterized by λ, so that eliminating λ and combining the result with the constraint is essentially equivalent to finding which member of the family satisfies the constraint.

EXAMPLE 5 John happens to acquire 420 feet of fencing and decides to use it to start a kennel by building 5 identical adjacent rectangular runs (see diagram below). Find the dimensions of each run that maximizes its area.



Solution: We let A denote the area of a run, and we let x, y be the dimensions of each run. Clearly, there are to be 10 sections of fence corresponding to widths x and 6 sections of fence corresponding to lengths y. Thus, we desire to maximize $A = xy$ subject to the constraint

$$10x + 6y = 420$$

Since x and y cannot be negative, we need only find absolute extrema for x in $[0, 42]$.
The Lagrangian for the problem is

$$L(x,y,\lambda) = xy - \lambda(10x + 6y - 420)$$

and $L_x = y - 10\lambda$, $L_y = x - 6\lambda$, and $L_\lambda = 10x + 6y - 420$. Thus, the critical points of $L(x,y,\lambda)$ satisfy

$$y = 10\lambda, \quad x = 6\lambda, \qquad 10x + 6y = 420$$

The first two equations parameterize the extrema in the parameter λ, which is why we eliminate λ to obtain $\lambda = y/10$ and $\lambda = x/6$. Thus,

$$\frac{y}{10} = \frac{x}{6}, \qquad y = \frac{10x}{6} = \frac{5x}{3}$$

Substituting into the constraint thus yields

$$10x + 6\left(\frac{5x}{3}\right) = 420, \qquad x = 21\ feet$$

Moreover, we also have $y = 5 \cdot 21/3 = 5 \cdot 7 = 35$ feet. At $x = 0$ and $x = 42$, the area is 0, while at the critical point $(20, 35)$, the area is 700 square feet. Thus, the maximum occurs when $x = 21$ feet and $y = 35$ feet.

If possible, a good approach to eliminating λ in a system of equations of the form

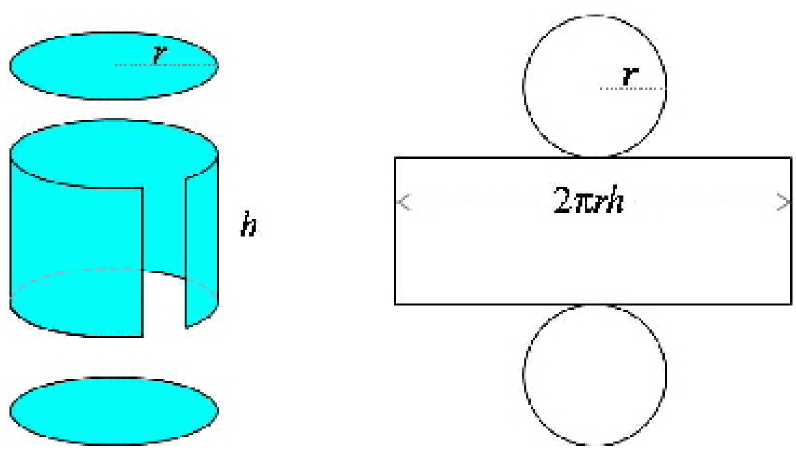
$$f_x = \lambda g_x, \quad f_y = \lambda g_y$$

is that of dividing the former by the latter to obtain

$$\frac{f_x}{f_y} = \frac{\lambda g_x}{\lambda g_y} \quad \Longrightarrow \quad \frac{f_x}{f_y} = \frac{g_x}{g_y}$$

and then cross-multiplying to obtain $f_x g_y = f_y g_x$. However, this method is not possible if one or more of the factors is zero.

EXAMPLE 6 A right cylindrical can is to have a volume of 0.25 cubic feet (approximately 2 gallons). Find the height h and radius r that will minimize surface area of the can. What is the relationship between the resulting r and h?



Solution: The surface area S is the sum of the areas of 2 circles of radius r and a rectangle with height h and width $2\pi r$. Thus,

$$S = 2\pi r^2 + 2\pi rh$$

This is constrained by a volume of $\pi r^2 h = 0.25\ ft^3$, so that the Lagrangian is

$$L(r,h,\lambda) = 2\pi r^2 + 2\pi rh - \lambda\left(\pi r^2 h - 0.25\right)$$

Setting $L_r = 0$ and $L_h = 0$ leads to

$$4\pi r + 2\pi h = \lambda(2\pi rh), \quad 2\pi r = \lambda\left(\pi r^2\right)$$

The ratio of the two equations is

$$\frac{4\pi r + 2\pi h}{2\pi r} = \frac{\lambda(2\pi rh)}{\lambda(\pi r^2)} \qquad \Longrightarrow \qquad \frac{2r+h}{r} = \frac{2h}{r}$$

Cross-multiplying yields $2r^2 + rh = 2rh$, which in turn yields

$$rh = 2r^2 \quad or \quad h = 2r$$

since r cannot be 0. That is, all cans with minimal surface area will have $h = 2r$, which means a height equal to the diameter. To determine which such can satisfies the constraint, we substitute to obtain

$$\pi r^2(2r) = 0.25, \quad r^3 = \sqrt[3]{\frac{0.25}{2\pi}}$$

which leads to $r = 0.3414$ feet, with $h = 0.6818$ feet. To see that a minimum must occur, we notice that the constraint implies that $h = 0.25/\left(\pi r^2\right)$, which leads to S as a function of r in the form

$$S = 2\pi r^2 + \frac{0.5}{r^2}$$

Straightforward differentiation shows that $S''(r) > 0$ for all $r > 0$, so that any extremum must be a minimum.

Check Your Reading: What is the value of the Lagrange multiplier in example 5?

Multiple Constraints

Typically, if given a constraint of the form $g(x,y) = k$, we instead let $g_1(x,y) = g(x,y) - k$ and use the constraint $g_1(x,y) = 0$. Thus, Lagrangians are usually of the form

$$L(x,y,z,\lambda) = f(x,y,z) - \lambda g_1(x,y,z)$$

Correspondingly, to find the extrema of a function $f(x,y,z)$ subject to *two* constraints,

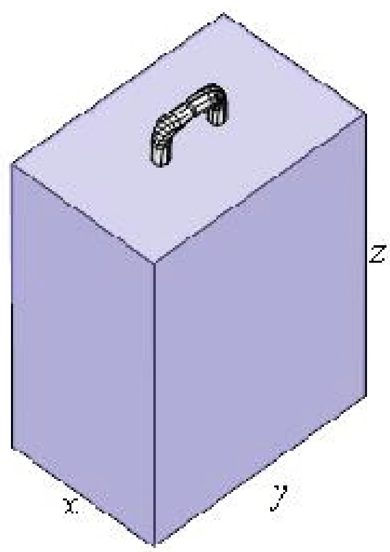
$$g(x,y,z) = k, \qquad h(x,y,z) = l$$

we define a function of the 3 variables $x, y,$and z, and the Lagrange multipliers λ and μ by

$$L(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda g_1(x,y,z) - \mu h_1(x,y,z)$$

where $g_1(x,y,z) = g(x,y,z) - k$ and $h_1(x,y,z) = h(x,y,z) - l$. As before, the goal is to determine the critical points of the Lagrangian.

blue*EXAMPLE 7* blackMany airlines require that carry-on luggage have a linear distance (sum of length, width, height) of no more than 45 inches with an additional requirement of being able to slide under the seat in front of you.



If we assume that the carry-on is to have (at least roughly) the shape of a rectangular box and one dimension is no more than half of one of the other dimensions (to insure "slide under seat" is possible), then what dimensions of the carryon lead to maximum storage (i.e., maximum volume)?

Solution: If we let x, y, and z denote length, width, and height, respectively, then our goal is to maximize the volume $V(x,y,z)$ subject to the constraints

$$x + y + z = 45 \quad and \quad y = 2x$$

(i.e., x is 1/2 of y). Thus, $g_1(x,y,z) = x + y + z - 45$ and $h_1(x,y,z) = y - 2x$ leads to a Lagrangian of the form

$$\begin{aligned} L(x,y,z,\lambda,\mu) &= f(x,y,z,) - \lambda g_1(x,y,z,) - \mu h_1(x,y,z,) \\ &= xyz - \lambda(x+y+z-45) - \mu(y-2x) \end{aligned}$$

The partial derivatives of L are

$$\begin{aligned} L_x &= yz - \lambda - \mu(-2), \quad L_y = xz - \lambda - \mu \\ L_z &= xy - \lambda \end{aligned}$$

and $L_\lambda = x + y + z - 45$, $L_\mu = y - 2x$. The critical points thus must satisfy

$$yz = \lambda - 2\mu, \quad xz = \lambda + \mu, \quad xy = \lambda$$

along with the constraints. Combining the last two equations yields $xz = xy + \mu$, so that the first equation becomes

$$yz = xy - 2(xz - xy) \quad or \quad yz = 3xy - 2xz$$

Since $y = 2x$, this becomes

$$2xz = 6x^2 - 2xz \quad or \quad 4xz = 6x^2$$

Since $x = 0$ leads to a zero volume, we must have $2z = 3x$, or $z = 1.5x$. Substituting into the first constraint yields

$$x + 2x + 1.5x = 45$$

which is $4.5x = 45$ or $x = 10$. If $x = 10$, then $y = 2x = 20$ and $z = 1.5x = 15$, so that the critical point is $(10, 20, 15)$. Since x, y, and z must all be in $[0, 45]$, we are seeking the extrema of the volume over a closed set (in particular, the closed box $[0,45] \times [0,45] \times [0,45]$) and the volume is zero on the boundary. Thus, the maximum volume must occur, and the only place left for it to occur is at the critical point $(10, 20, 15)$.

Of course, we could substituted $y = 2x$ directly to reduce example 7 to an ordinary Lagrange multiplier problem:

$$\begin{aligned} Maximize\ Volume &: \quad V = 2x^2 z \\ Subject\ to &: \quad x + 2x + z = 45 \end{aligned}$$

However, not all multiple constraint problems share this feature. Moreover, example 7 illustrates how the Lagrange multiplier method can be applied to optimizing a function f of any number of variables subject to any given collection of constraints. Specifically, as is shown in the accompanying worksheet, the associated Lagrangian has a multiplier term corresponding to each constraint.

Exercises

Use the method of Lagrange Multipliers to find the extrema of the following functions subject to the given constraints. (Notice that in each problem below the constraint is a closed curve).

1. $f(x, y) = 3x + 2y$ subject to: $x^2 + y^2 = 1$
2. $f(x, y) = 2x - y$ subject to: $x^2 + y^2 = 1$
3. $f(x, y) = x - 2y$ subject to: $x^2 + y^2 = 25$
4. $f(x, y) = x + y$ subject to: $x^2 + 2y^2 = 1$
5. $f(x, y) = x^2 + 2y^2$ subject to: $x^2 + y^2 = 1$
6. $f(x, y) = x^2 y$ subject to: $x^2 + y^2 = 1$
7. $f(x, y) = x^4 + y^2$ subject to: $x^2 + y^2 = 1$
8. $f(x, y) = x^4 + y^4$ subject to: $x^2 + y^2 = 1$
9. $f(x, y) = x \sin(y)$ subject to: $x^2 + y^2 = 1$
10. $f(x, y) = \sin(x) \cos(y)$ subject to: $x^2 + y^2 = 1$

Find the point(s) on the given curve closest to the origin.

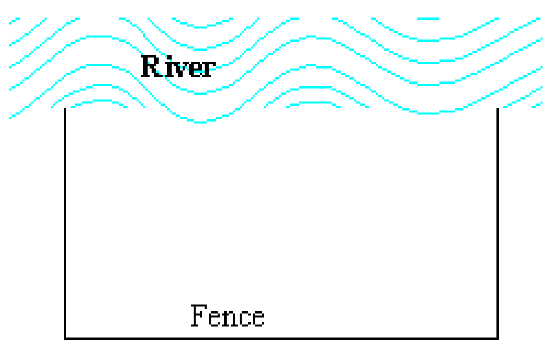
11. $x + y = 1$
12. $x + 2y = 5$
13. $x = y^2 - 1$
14. $x^2 + 2y^2 = 1$
15. $y = e^{-x^2/2+2}$
16. $y = 2e^{-x^2/2}$

Use the method of Lagrange multipliers in problems 17-30

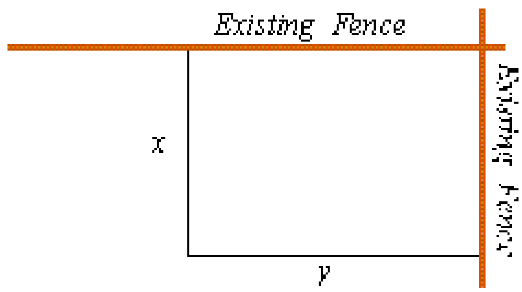
17. Maximize the product of two positive numbers whose sum is 36.

18. Minimize the sum of two positive numbers whose product is 36.

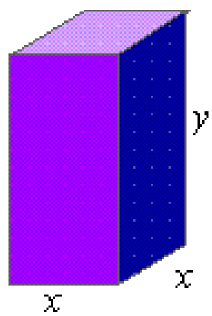
19. A farmer has 400 feet of fence with which to enclose a rectangular field bordering a river. What dimensions of the field maximize the area if the field is to be fenced on only three sides (see picture below)?



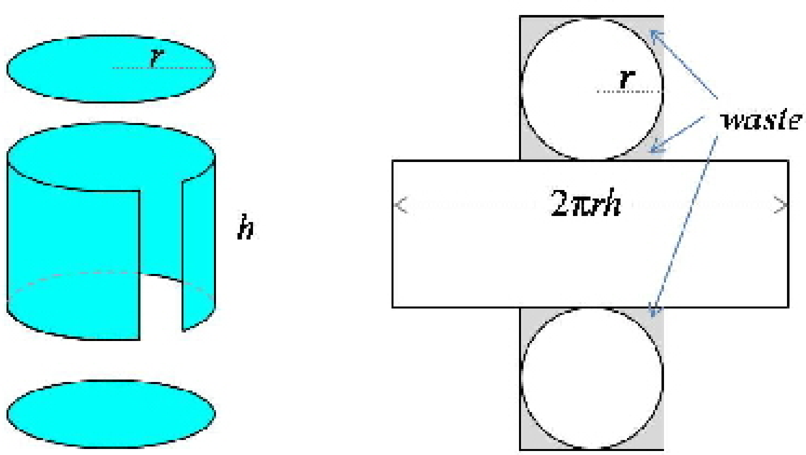
20. A farmer has 400 feet of fence with which to fence in a rectangular field adjoining two existing fences which meet at a right angle. What dimensions maximize the area of the field?



21. A rectangular box with a square bottom is to have a volume of 1000 ft^3. What dimensions for the box yield the smallest surface area?



22. A soupcan can is to have a volume of 25 cubic inches, and it is to be made with as little metal as possible. The manufacturing process makes the cans by rolling up rectangles of metal and capping each end with circles *punched from a square whose sides are the lengths of the diameters of the top and bottom.*



This means minimizing the surface area of the can *as well as the "wasted area" between the ends and the squares they are punched from.* Use Lagrange multipliers to find r and h that minimize the metal used to make the can. What is the relationship between the resulting r and h?

23. What dimensions of the carry on in example 7 yield maximum volume if we drop the requirement of one dimension being no more than half of another.

24. What dimensions of the carry on in example 7 yield maximum volume if we drop the requirement of one dimension being no more than half the other and *set* $z = 9$ (i.e., find x and y).

25. Redo example 7 for those airlines that allow 51 linear inches for a carryon.

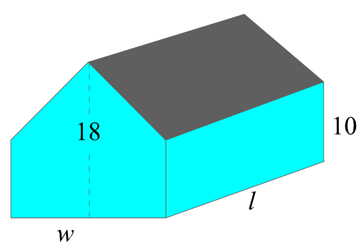
26. What dimensions of the carry on in example 7 yield maximum volume if we replace the constraint $y = 2x$ by the constraint $xy = 300$? What does this new constraint represent?

27.

27. The intersection of the plane $z = 4.5 + 0.5x$ and the cone $x^2 + y^2 = z^2$ is an ellipse. What point(s) on the ellipse are furthest from or closest to the origin? What is significant about these points?

28. The intersection of the plane $z = 1 + x$ and the cone $x^2 + y^2 = z^2$ is a parabola. What point on the parabola is closes to the origin? What is significant about these point?

29. A house with width w and length l is be 18 feet tall at its tallest and 10 feet tall at each corner.



What dimensions for a 2000 square foot house (i.e., $wl = 2000$) minimize the area of the roof and sides of the house?

30. The moon's orbit about the earth is well-approximated by the curve

$$x^2 + y^2 = (238,957 - 0.0549y)^2$$

where distance is in miles. How close is the moon to the earth at its closest point? What is the greatest distance between the moon and the earth?

Optimization with constraints occurs frequently in business settings. For example, if L denotes the number of manhours and K denotes the number of units of capital required to produce q units of a commodity, then q is often related to L and K by a Cobb-Douglas function

$$q = AL^{\alpha}K^{\beta} \tag{4}$$

where A is a constant, α is the product elasticity of labor, and β is the product elasticity of capital. If one unit of labor costs w dollars and one unit of capital costs r dollars, then the cost to manufacture q units of a commodity is

$$C = wL + rK \tag{5}$$

Often the number of items to be produced, q, is a constant, so that (4) is used as a constraint to (5). Use the Cobb-Douglas production model in exercises 31-34 to find the relationship between L and K that minimizes total cost.

31. Cobb and Douglas introduced the **idealized production function**

$$q = AL^{3/4}K^{1/4}$$

as a model of the interplay of Labor and Capital in the U.S. economy from 1889 to 1929. Find the values of L and K that minimize total cost and determine the **ratio** of labor to capital when total cost is minimized, given that $A = 0.8372$ during that time and that one unit of capital has the same cost as one unit of capital.

32. Erwin is contracted to produce 50 corner cabinets. He knows he will need help (labor), and he knows that he must expand his basement workshop (capital). If Erwin estimates his production will satisfy

$$50 = 0.4L^{0.8}K^{0.4}$$

then what amount of labor and capital will minimize the cost to produce the cabinets when labor is \$20 per hour and capital is \$10 per unit (A unit of capital in Erwin's case may be considered the amount to increase his workshop by the equivalent of one handtool.)?

33. A firm produces a commodity with a Cobb-Douglas production model of $q = 4L^{2/3}K^{1/3}$, where labor costs $w = \$15$ per manhour and capital costs $r = \$10$ per unit. What values of L and K minimize the cost of producing $q = 1000$ units of the commodity?

34. Suppose in exercise 33 that the firm has only \$10,000 to spend on producing the commodity. What values of L and K will maximize the output, q?

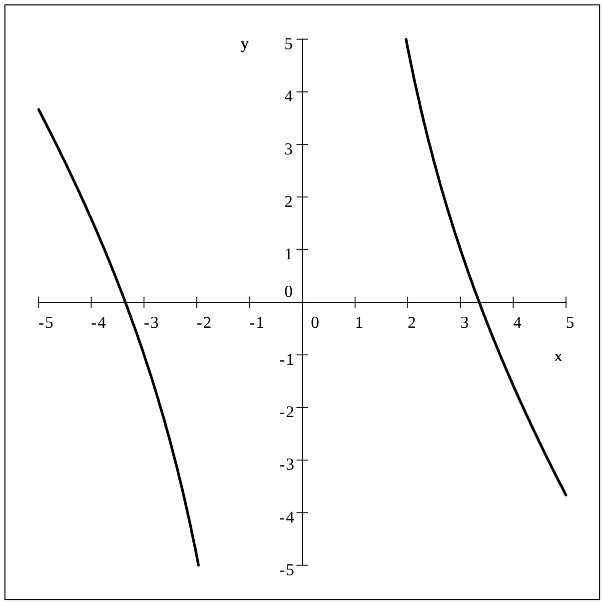
35. Find the extrema of $f(x, y, z) = x + yz$ subject to the constraints

$$x^2 + y^2 + z^2 = 1, \ z^2 = x^2 + y^2$$

36. Find the extrema of $f(x, y, z) = xyz$ subject to the constraints

$$x^2 + y^2 + z^2 = 1 \quad \textit{and} \quad xy + yz + zx = 1$$

37. Find the point P on the curve $4x^2 + 3xy = 45$ that is closest to the origin, and then show that the line from the origin through that point P is perpendicular to the tangent line to the curve at P. In light of the method of Lagrange multipliers, why would we expect this result? (Hint: you may prefer to eliminate λ using the method described between examples 5 and 6).



38. Find the point P on the curve $y = x^2 + x - 1.5$ that is closest to the origin, and then show that the line from the origin through that point P is perpendicular to the tangent line to the curve at p. In light of the method of Lagrange multipliers, why would we expect this result?

39. Find the maximum of

$$f(x,y) = e^{-x^2-y^2}$$

subject to the constraint $x + y = 1$. Explain why we know that we have a maximum even though the constraint is not a closed curve.

40. Find the maximum of

$$f(x,y) = e^{-x^2-y^2}$$

subject to the constraint $x^2 - y^2 = 1$. Explain why we know that we have a maximum even though the constraint is not a closed curve.

41. Show that ∇f is parallel to ∇g only if

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} = 0 \tag{6}$$

Then use (6) to find the extrema of $f(x,y) = 3x + 4y$ subject to $x^2 + y^2 = 25$.

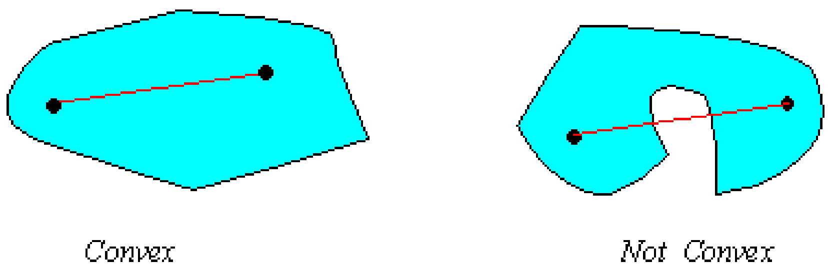
42. Use the method in problem 39 to find the extrema of $f(x,y) = xy$ subject to $x^2 + 2y^2 = 5$.

43. Write to Learn: Write a short essay in which you explain why if $g(x,y) = k$ is a smooth closed curve, then a continuous function $f(x,y)$ must attain its maximum and minimum values at solutions to the Lagrange multiplier problem. (Hint: if $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, t in $[a,b]$, is a parametrization of $g(x,y) = k$, then finding the extrema of $f(x,y)$ subject to $g(x,y)=k$ is the same as finding the extrema of

$$z(t)=f(x(t),y(t))$$

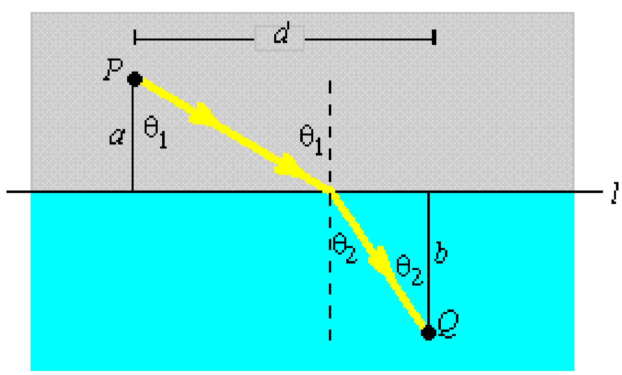
when $z(a)=z(b)$.)

44. Write to Learn: A region in the xy-plane is said to be *convex* if the line segment joining any two points in the region is also in the region.



Show that if R is a closed convex region and if $f(x,y)=mx+ny$ is a linear function (i.e., m,n constant), then the largest possible value of $f(x,y)$ over the region R must occur on the boundary of R. (Hint: let $g(x,y)=k$ be any line connecting two boundary points of R)

45. Write to Learn: Suppose that light travels from a point P with a constant speed v_1 in the medium above a horizontal line l and suppose that it travels to Q with a constant speed v_2 in the medium below l.



In a short essay, use Lagrange multipliers to explain why that the angles θ_1 and θ_2 that minimize the time required to travel from P to Q must satisfy

$$\frac{\sin(\theta_1)}{v_1}=\frac{\sin(\theta_2)}{v_2} \tag{7}$$

(Hint: Constraint is that horizontal distance traveled is equal to a fixed distance d).

Maple Extra: Create a worksheet which demonstrates Snell's law (7) using animation and through calculation of time of travel for different values of θ_1 and θ_2.