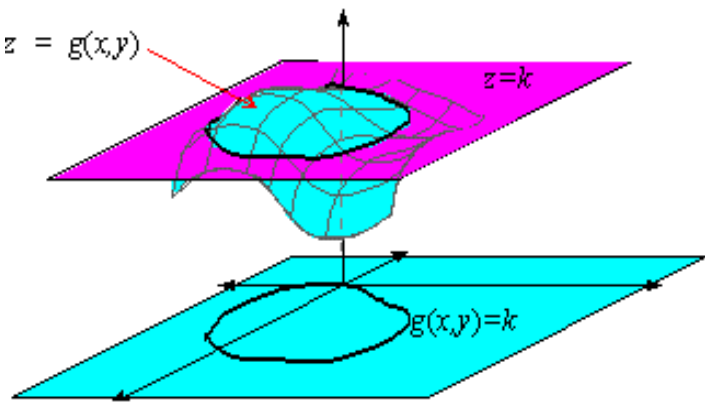
Properties of the Gradient

Gradients and Level Curves

In this section, we use the gradient and the chain rule to investigate horizontal and vertical slices of a surface of the form $z = g(x, y)$. To begin with, if k is constant, then $g(x, y) = k$ is called the *level curve* of $g(x, y)$ of level k and is the intersection of the horizontal plane $z = k$ and the surface $z = g(x, y)$:



In particular, $g(x, y) = k$ is a curve in the xy-plane.

EXAMPLE 1 Construct the level curves of $g(x, y) = x^2 + y^2 + 2$ for $k = 2, 3, 4, 5$, and 6.

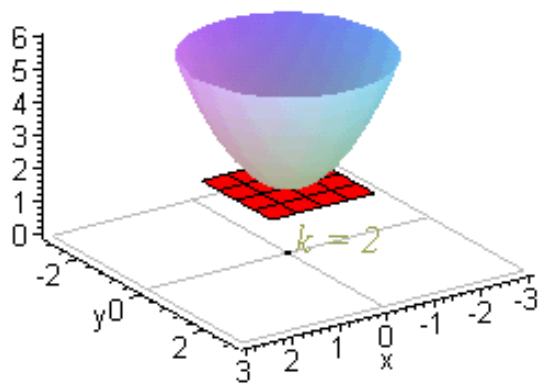
Solution: To begin with, $g(x, y) = k$ is of the form $x^2 + y^2 + 2 = k$, which simplifies to

$$x^2 + y^2 = k - 2$$

If $k = 2$, then $x^2 + y^2 = 0$ which is true only if $(x, y) = (0, 0)$. For $k = 3, 4, 5$,and 6, we have

$$\begin{array}{llll} k = 3: & x^2 + y^2 = 1 & k = 5: & x^2 + y^2 = 3 \\ k = 4: & x^2 + y^2 = 2 & k = 6: & x^2 + y^2 = 4 \end{array}$$

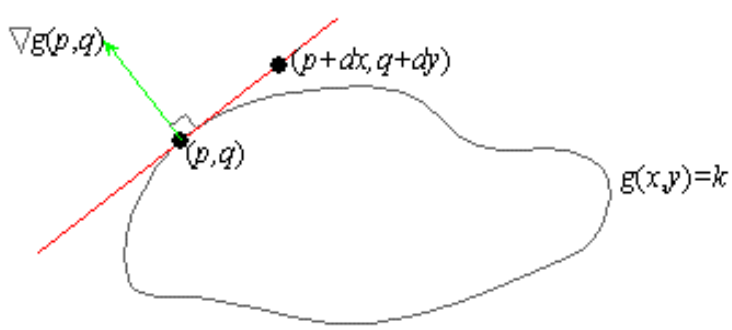
These are circles centered at $(0,0)$ with radii $1, \sqrt{2}, \sqrt{3}$, and 2, respectively.



If $g(x,y)$ is differentiable, then the level curve $g(x,y)=k$ can be parametrized by a vector-valued function of the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. If $\mathbf{v}$ denotes the velocity of $\mathbf{r}$, then the chain rule implies that

$$\frac{d}{dt}g(x,y) = \frac{d}{dt}k \quad \Longrightarrow \quad \nabla g \cdot \mathbf{v} = 0$$

Thus, ∇g is *perpendicular* to the tangent line to $g(x,y)=k$ at the point (p,q).



We say that $\nabla g(p,q)$ is *normal* to the curve $g(x,y)=k$ at (p,q).

EXAMPLE 2 Show that the gradient is normal to the curve $y = 1-2x^2$ at the point $(1,-1)$.

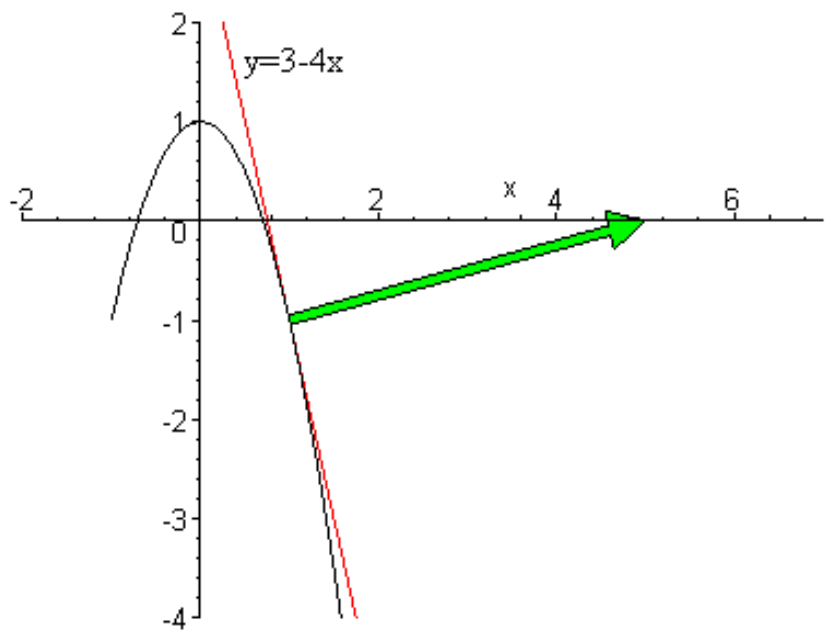
Solution: To do so, we notice that $2x^2 + y = 1$. Thus, the curve is of the form $g(x, y) = 1$ where $g(x, y) = 2x^2 + y$. The gradient of g is

$$\nabla g = \langle 4x, 1 \rangle$$

Thus, at $(1, -1)$, we have $\nabla g(1, -1) = \langle 4, 1 \rangle$. However, if $y = 1 - 2x^2$, then $y' = -4x$, and when $x = 1$, we have $y'(1) = -4$. That is, a run of 1 leads to a rise of -4, so that the slope corresponds to the vector

$$\mathbf{m} = \langle run, rise \rangle = \langle 1, -4 \rangle$$

Clearly, $\nabla g \cdot \mathbf{m} = \langle 4, 1 \rangle \cdot \langle 1, -4 \rangle = 0$, thus implying that ∇g is perpendicular to the tangent line to $y = 1 - 2x^2$ at $(-1, 1)$.



Check Your Reading: Are there any level curves of $g(x, y) = x^2 + y^2 + 2$ with a level lower than $k = 2$?

The Directional Derivative

If $f(x, y)$ is differentiable at a point $\mathbf{p} = (p, q)$ and if $\mathbf{u} = \langle m, n \rangle$ is a unit vector, then the *derivative of* f at $\mathbf{p}$ in the *direction of* $\mathbf{u}$ is defined to be

$$(D_{\mathbf{u}} f)(\mathbf{p}) = \lim_{h \to 0^+} \frac{f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})}{h} \tag{1}$$

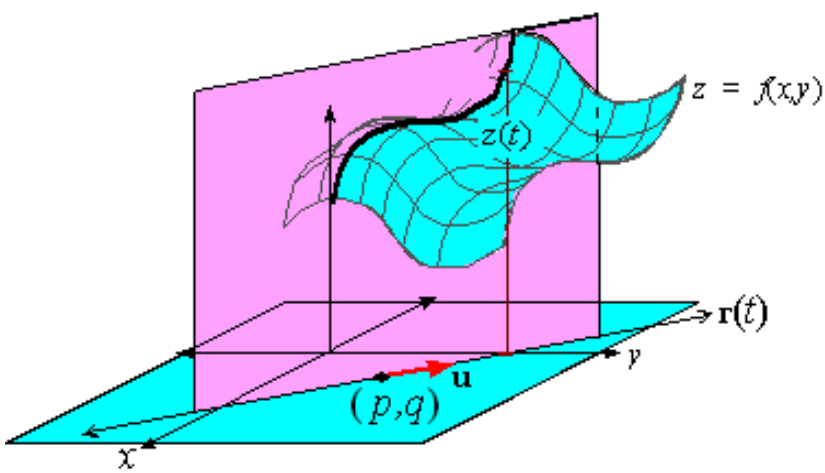
We say that $D_{\mathbf{u}}f$ is the *directional derivative* of f in the direction of $\mathbf{u}$. For example, in the direction of $\mathbf{i}$ we have

$$(D_{\mathbf{i}}f)(\mathbf{p}) = \lim_{h\to 0^+} \frac{f(\mathbf{p}+h\mathbf{i}) - f(\mathbf{p})}{h} = \lim_{h\to 0^+} \frac{f(p+h,q) - f(p,q)}{h} = f_x(p,q)$$

Indeed, the directional derivatives in the directions of $\mathbf{i}$ and $\mathbf{j}$, respectively, are the first partial derivatives

$$D_{\mathbf{i}}f = \frac{\partial f}{\partial x} \quad and \quad D_{\mathbf{j}}f = \frac{\partial f}{\partial y}$$

The directional derivative can be interpreted geometrically via *vertical slices* of the surface $z = f(x,y)$, where a vertical slice is a curve formed by the intersection of the surface $z = f(x,y)$ with the vertical plane through a line $\mathbf{r}(t) = \mathbf{p} + \mathbf{u}t$ in the xy-plane.

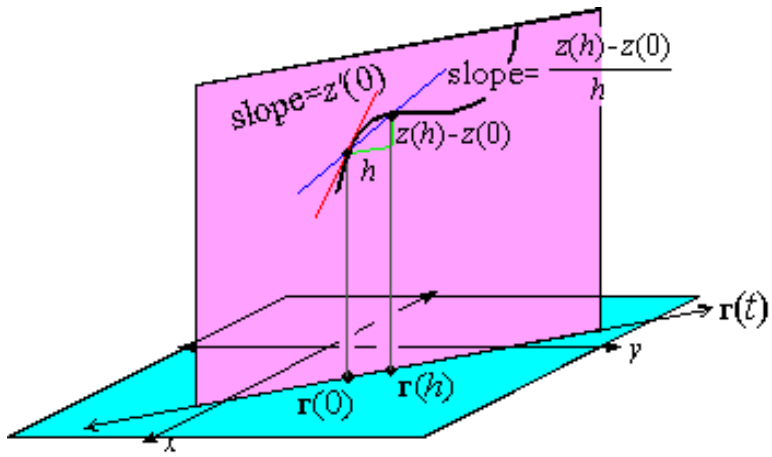


Specifically, the z-coordinate of the vertical slice is $z(t) = f(\mathbf{p} + t\mathbf{u})$, which in non-vector form is $z(t) = f(p+mt, q+nt)$. The definition of the directional derivative is equivalent to

$$(D_{\mathbf{u}}f)(\mathbf{p}) = z'(0)$$

Since $\mathbf{u}$ is a unit vector, the point $\mathbf{r}(h)$ is a distance h from $\mathbf{r}(0)$. Thus, a

"run" of h causes a "rise" of $z(h) - z(0)$



Thus, as h approaches 0, the slope of the secant line (in blue) approaches the slope of the tangent line (in red). That is, the slope of the tangent line at $\mathbf{r}(0) = \mathbf{p} = (p, q)$ is

$$z'(0) = \lim_{h \to 0} \frac{z(h) - z(0)}{h}$$

where $z(t) = f(p + mt, q + nt)$. Moreover, since $\mathbf{r}(t) = \mathbf{p} + \mathbf{u}t$ implies that $\mathbf{r}'(0) = \mathbf{u}$ and since $(D_{\mathbf{u}} f)(\mathbf{p}) = z'(0)$, the chain rule implies that

$$z'(0) = \left.\frac{dz}{dt}\right|_{t=0} = \nabla f \cdot \mathbf{r}'(0) = \nabla f \cdot \mathbf{u}$$

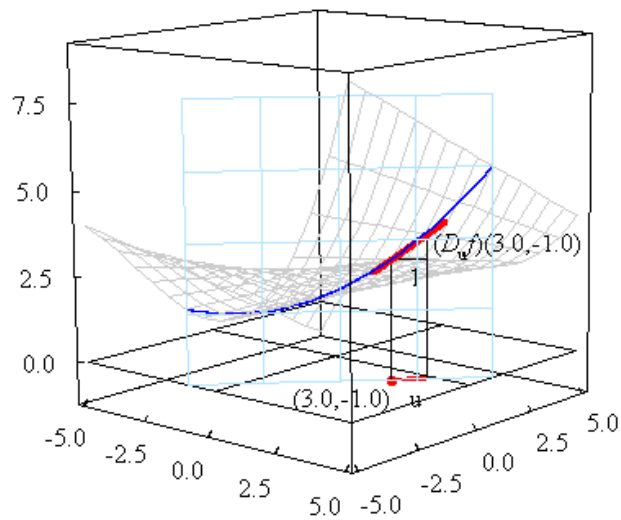
That is, the slope of the tangent line is $\nabla f \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction of $\mathbf{r}(t)$.

Theorem 7.1 The *directional derivative* of $f(x, y)$ in the direction of a unit vector $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

Moreover, $D_{\mathbf{u}} f$ is the slope of the tangent line to the curve formed by the intersection of $z = f(x, y)$ and the vertical plane through a

point $\mathbf{p}$ parallel to a unit vector $\mathbf{u}$.



Moreover, theorem 7.1 confirms what we alluded to in section 3, which is that f_x yields slopes of vertical slices parallel to the xz-plane and f_y yields slopes of vertical slices parallel to the yz-plane.

EXAMPLE 3 Find the derivative of $f(x,y) = 1.1x^2 - 0.1xy$ in the direction of $\mathbf{v} = \langle 3, 4 \rangle$.

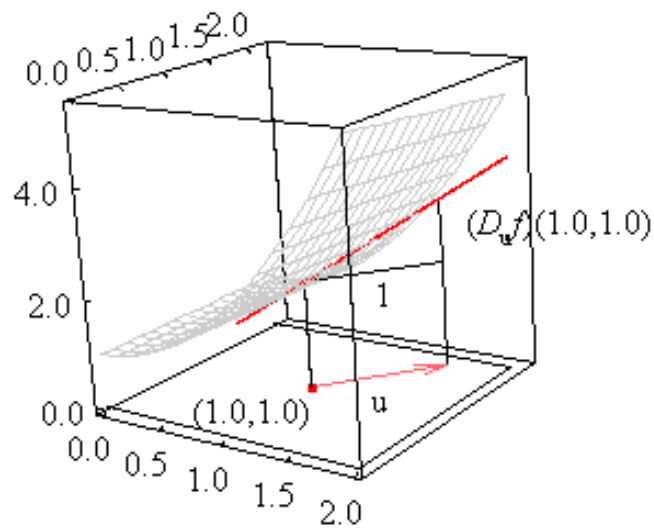
Solution: Since $\mathbf{v}$ is not a unit vector, we first finds its direction vector:

$$\mathbf{u} = \frac{1}{v}\mathbf{v} = \frac{1}{5}\langle 3, 4 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \langle 0.6, 0.8 \rangle$$

The gradient of f is $\nabla f = \langle 2.2x - 0.1y, -0.1x \rangle$, so that

$$\begin{aligned} D_{\mathbf{u}} f &= \langle 2.2x - 0.1y, -0.1x \rangle \cdot \langle 0.6, 0.8 \rangle \\ &= 0.6\,(2.2x - 0.1y) + 0.8\,(-0.1x) \\ &= 1.24x - 0.06y \end{aligned}$$

The applet below shows $D_{\mathbf{u}}f$ at the point $(1,1)$.



.

Check your Reading: What is $D_{\mathbf{u}}f$ at the point $(1,0)$?

Properties of the Gradient

If we let θ denote the angle between ∇f and a given unit vector $\mathbf{u}$, then

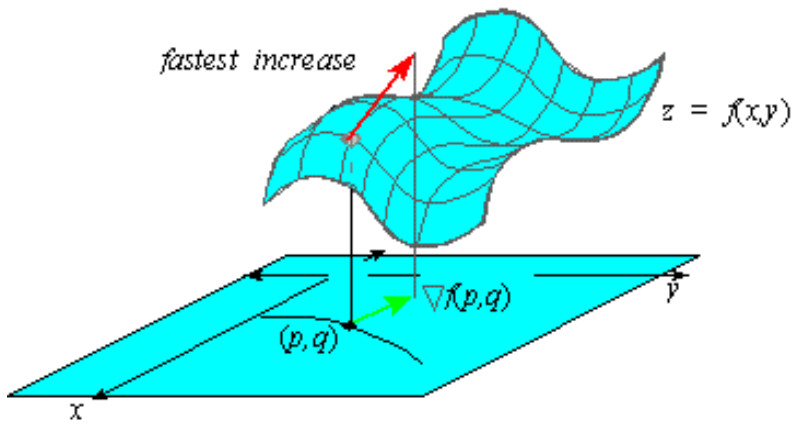
$$D_{\mathbf{u}}f = \|\nabla f\| \, \|\mathbf{u}\| \cos(\theta) = \|\nabla f\| \, \cos(\theta)$$

since $\|\mathbf{u}\| = 1$. Thus, the directional derivative is largest when $\theta = 0$, which is in the direction of ∇f, and is smallest when $\theta = \pi$, which is in the direction of $-\nabla f$.

> **Theorem 7.2:** The slope $(D_{\mathbf{u}}f)(\mathbf{p})$ of the tangent line to a vertical slice of $a = f(x,y)$ through $(p,q,f(\mathbf{p}))$ is *greatest* in the direction of $\nabla f(\mathbf{p})$ and is *smallest* in the direction of $-\nabla f(\mathbf{p})$

TWe say that $\nabla f(p,q)$ is the *direction of steepest ascent* for the surface $z = f(x,y)$ at (p,q). Analogously, $-\nabla f(p,q)$ must the *direction of steepest descent*

or *direction of least resistance* at the point (p, q).



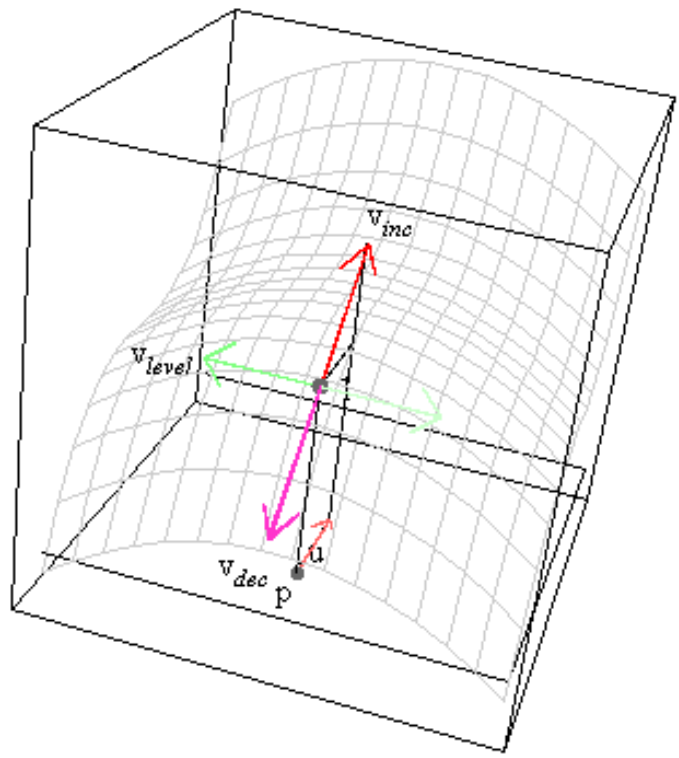
Moreover, when $\mathbf{u}$ is in the direction of ∇f, then $\mathbf{u} = \nabla f / \|\nabla f\|$, so that

$$D_{\mathbf{u}} f = \nabla f \cdot \frac{\nabla f}{\|\nabla f\|} = \frac{\|\nabla f\|^2}{\|\nabla f\|} = \|\nabla f\|$$

Similarly, if $\mathbf{u}$ is in the direction of $-\nabla f$, then $D_{\mathbf{u}} f = -\|\nabla f\|$. Also, the corresponding tangent vectors to the surface are

$$\begin{aligned} fastest\ increase \quad &: \quad \mathbf{v}_{inc} = \left\langle \frac{f_x}{\|\nabla f\|}, \frac{f_y}{\|\nabla f\|}, \|\nabla f\| \right\rangle \\ fastest\ decrease \quad &: \quad \mathbf{v}_{dec} = \left\langle \frac{-f_x}{\|\nabla f\|}, \frac{-f_y}{\|\nabla f\|}, \|\nabla f\| \right\rangle \end{aligned}$$

This is important in many numerical routines.



EXAMPLE 4 How fast is $f(x,y) = x^2 + y^3$ increasing at $(2,1)$ when it is increasing the fastest?

Solution: To begin with, $\nabla f = \langle 2x, 3y^2 \rangle$. Moreover, f is increasing the fastest when $\mathbf{u}$ is in the direction of $\nabla f(2,3) = \langle 4,3 \rangle$, and in this direction, we have

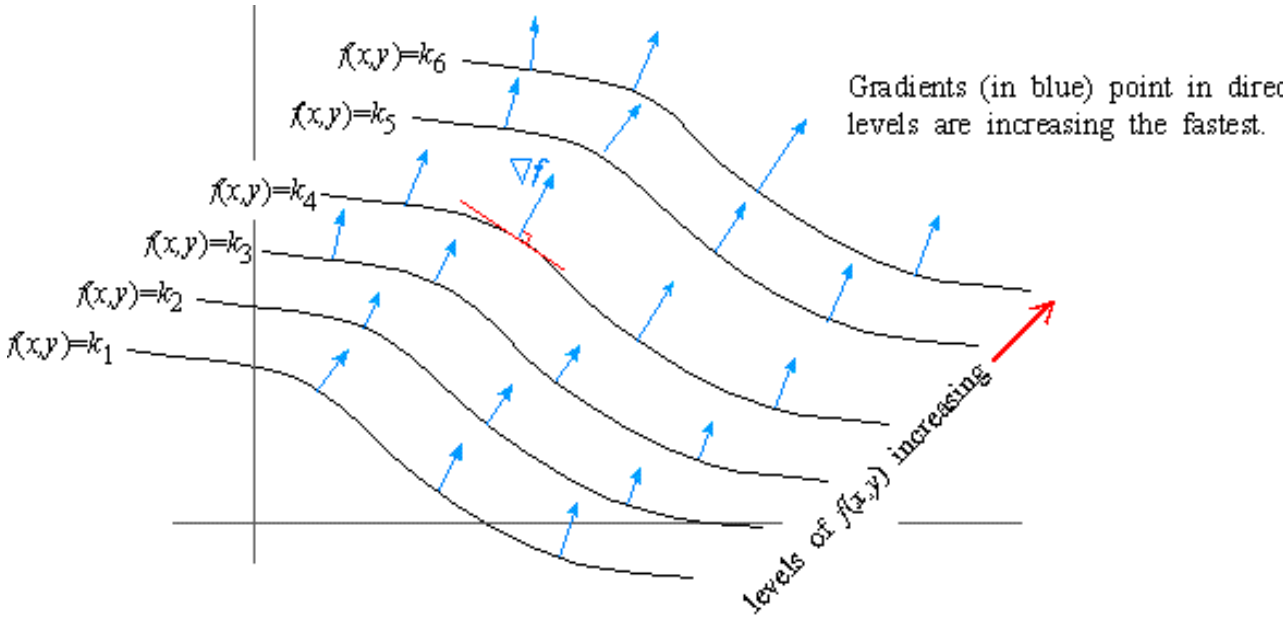
$$(D_{\mathbf{u}} f)(2,1) = \|\langle 4,3 \rangle\| = 5$$

(The notation $(D_{\mathbf{u}} f)(p,q)$ means the directional derivative $D_{\mathbf{u}} f$ evaluated at the point (p,q).)

Moreover, given a collection of increasing constants $k_1 < k_2 < k_3 < \ldots$, the set of level curves

$$f(x,y) = k_1, \quad f(x,y) = k_2, \quad f(x,y) = k_3, \quad \text{and so on}$$

forms a family of level curves in the xy-plane in which the levels are increasing. It then follows that $\nabla f(p,q)$ points in the direction in which the levels are increasing the fastest, and $-\nabla f(p,q)$ points in the direction in which the levels are decreasing the fastest.



EXAMPLE 5 In what direction are the levels of $f(x,y) = x^3y$ increasing the fastest at the point $(2,3)$?

Solution: The levels of f are increasing the fastest in the direction of $\nabla f(2,3)$. Since $\nabla f = \langle 3x^2y, x^3 \rangle$, we have $\nabla f(2,3) = \langle 36, 8 \rangle$. Thus, the levels of f are increasing the fastest in the direction

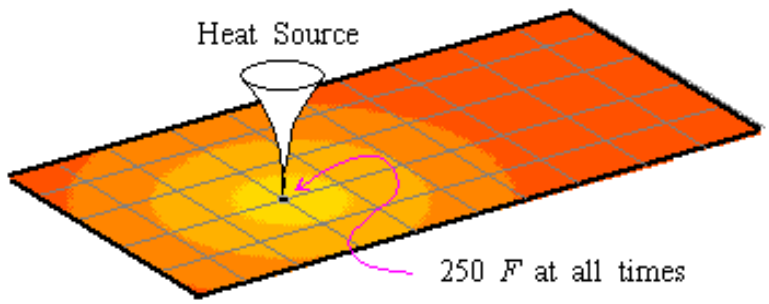
$$\mathbf{u} = \frac{\nabla f(2,3)}{\|\nabla f(2,3)\|} = \frac{\langle 36,8 \rangle}{\|\langle 36,8 \rangle\|} = \left\langle \frac{9}{\sqrt{85}}, \frac{2}{\sqrt{85}} \right\rangle$$

Check Your Reading: How fast are the levels increasing at right angles to ∇g?

Scalar Fields

Often functions of 2 variables are interpreted to be *scalar fields,* where a scalar field is the assignment of a scalar to each point in the plane. For example, let's

suppose that a $10'' \times 5''$ rectangular plate is at room temperature of $70°F$ when a heat source at a temperature of $250°F$ is applied at a point $3''$ from a shorter side and $2''$ from a longer side of the plate (i.e., at the point $(3, 2)$ if $(0, 0)$ is at a corner).



Suppose now that 5 seconds after the heat source is applied, each point (x, y) on the plate's surface then has a unique temperature, $T = g(x, y)$ where

$$g(x, y) = 70 + 180e^{-(x-3)^2/10-(y-2)^2/10} \tag{2}$$

That is, $g(x, y)$ is the scalar field which maps a point (x, y) to the temperature T of the surface at that point after 5 seconds.

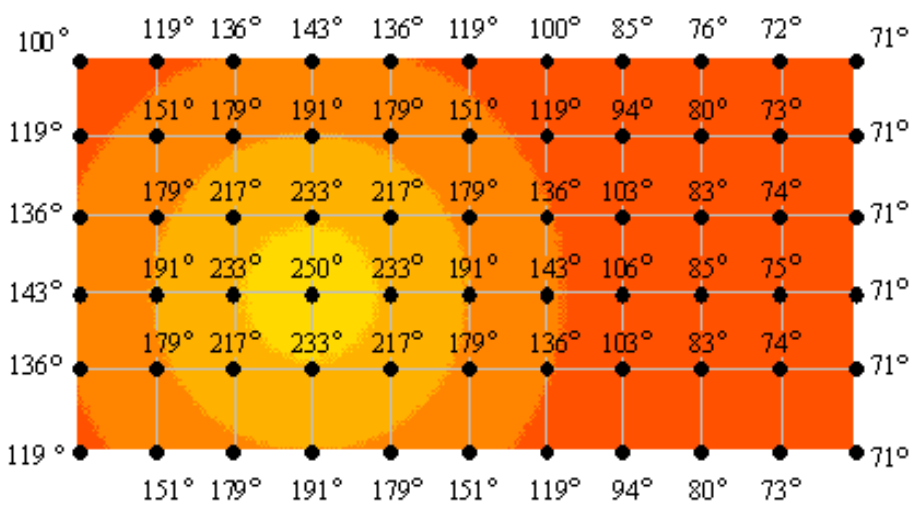
It follows that the temperature at the origin is

$$T = 70 + 180e^{-(0-3)^2/10-(0-2)^2/10} = 119°F$$

and that the temperature at the point $(1, 1)$ is

$$T = 70 + 180e^{-(1-3)^2/10-(1-2)^2/10} = 179°F$$

Temperatures at all integer pairs (x, y) in $[0, 10] \times [0, 5]$ are shown below:



(3)

If we now compute the gradient of g, we obtain

$$\nabla g = 180e^{-(x-3)^2/10-(y-2)^2/10}\left\langle -\frac{1}{5}x+\frac{3}{5}, -\frac{1}{5}y+\frac{2}{5}\right\rangle \tag{4}$$

At the point $(1,1)$, we then have

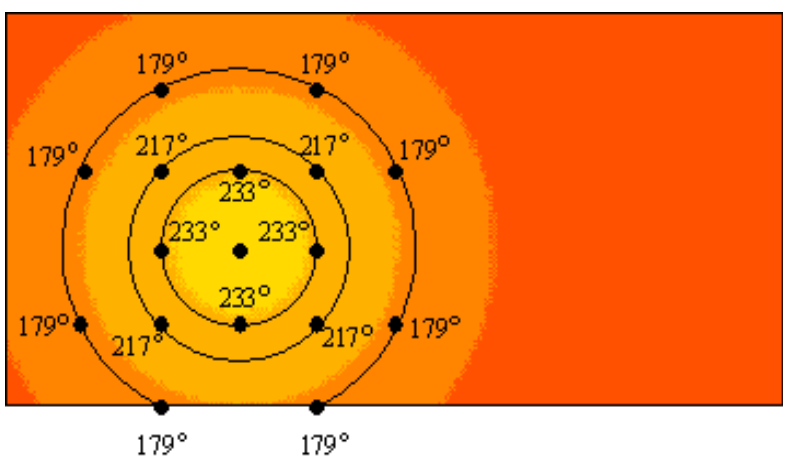
$$\begin{aligned}\nabla g &= 180e^{-(1-3)^2/10-(1-2)^2/10}\left\langle -\frac{1}{5}+\frac{3}{5}, -\frac{1}{5}+\frac{2}{5}\right\rangle \\ &= 109.17\,\langle 0.2, 0.1\rangle \\ &= \langle 43.668, 21.834\rangle\end{aligned}$$

This defines the direction in which the temperature is increasing the fastest. Moreover,

$$D_{\mathbf{u}}g = \parallel \langle 43.668, 21.834\rangle \parallel = 48.82^\circ F \textit{ per inch}$$

is the rate of increase in the direction of fastest increase.

Level curves of scalar fields are often referred to using prefixes such as "iso-" or "equi-". For example, the level curves of a scalar temperature field are known as *isotherms.* Below are shown the isotherms of the temperature field (2) for temperatures $179^\circ F$, $217^\circ F$, and $233^\circ F$



(5)

We'll explore more applications of scalar fields in the exercises.

Exercises

Sketch the level curves of the following functions at the given levels.

1.	$f(x,y) = x^2 + y^2$ $k = 1, 4, 9, 16$	2.	$f(x,y) = y - x^2$ $k = 0, 1, 2, 3, 4$
3.	$f(x,y) = xy$ $k = 1, 2, 3, 4$	4.	$f(x,y) = x^2 + 4y^2$ $k = 0, 4, 16, 100$
5.	$f(x,y) = (x-y)^2$ $k = 1, 4, 9, 16$	6.	$f(x,y) = xy$ $k = -2, -1, 0, 1, 2$

Find the gradient of the function implied by the level curve, and then show that it is perpendicular to the tangent line to the curve at the given point (in 15-17, you will need to use implicit differentiation to find the slope of the tangent line).

7.	$y = x^3 + x$, at $(1,2)$	8.	$y = x^2$, at $(2,4)$
9.	$x^2 + y = 2$, at $(1,1)$	10.	$x^2 + xy = 10$, at $(2,3)$
11.	$x - y = 1$ at $(2,1)$	12.	$x + y = 5$, at $(1,-2)$
13.	$xy = 1$, at $(1,1)$	14.	$x^2y = 2$ at $(1,2)$
15.	$x\sin(xy) = 0$, at $(1,\pi)$	16.	$x\cos\left(x^2y\right) = 2$, at $(2,\pi)$
17.	$2e^x + ye^x = y^2$, at $(0,2)$	18.	$\sin\left(x^2y\right) = x + 1$, at $(-1,\pi)$

Find the directional derivative of f in the direction of the given vector at the given point.

19.	$f(x,y) = x^3 + y^3$ at $(1,1)$ in direction of $\mathbf{v} = \langle 2,1\rangle$	18.	$f(x,y) = x^4 + y^2$ at $(1,1)$ in direction of $\mathbf{v} = \langle -3,4\rangle$
21.	$f(x,y) = \cos(xy)$ at $(\pi/2,0)$ in direction of $\mathbf{v} = \langle 1,0\rangle$	20.	$f(x,y) = \cos(xy)$ at $(\pi/2,0)$ in direction of $\mathbf{v} = \langle 2,1\rangle$
23.	$f(x,y) = xy$ at $(1,1)$ in direction of $\mathbf{v} = \langle 1,1\rangle$	24.	$f(x,y) = x\sin(xy)$ at $(\pi,0)$ in direction of $\mathbf{v} = \langle 1,-1\rangle$

Find the direction of fastest increase of the function at the given point, and then find the rate of change of the function when it is changing the fastest at the given point.

25.	$f(x,y) = x^3 + y^3$ at $(1,1)$	26.	$f(x,y) = x^4 + y^2$ at $(1,1)$
27.	$f(x,y) = \cos(xy)$ at $(\pi/2,0)$	28.	$f(x,y) = \cos(xy)$ at $(\pi/2,0)$
29.	$f(x,y) = xy$ at $(1,1)$	30.	$f(x,y) = x\sin(xy)$ at $(\pi,0)$

31. Show that $(1,2)$ is a point on the curve $g(x,y) = 2$ where $g(x,y) = x^{-2}y$. Then find the slope of the tangent line to $g(x,y) = 2$ in two ways:

1. (a) By finding using the gradient to find slope.
   (b) By showing that $y = 2x^2$ and applying the ordinary derivative.

Then sketch the graph of the curve, the tangent line, and the gradient at that point.

32. Show that $(3,2)$ is a point on the curve $g(x,y) = 15$ where $g(x,y) = x^2 + xy$. Then find the slope of the tangent line to $g(x,y) = 15$ in two ways:

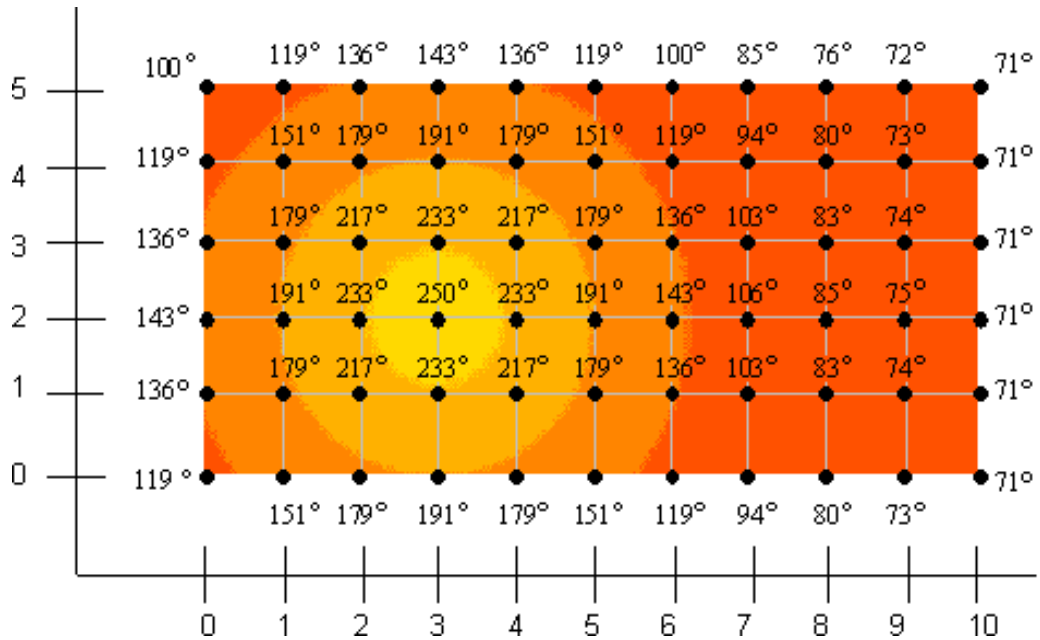
1. (a) By finding using the gradient to find slope.
   (b) By solving for y and applying the ordinary derivative.

Then sketch the graph of the curve, the tangent line, and the gradient at that point.

Exercises 33 - 36 continue the exploration of the scalar field

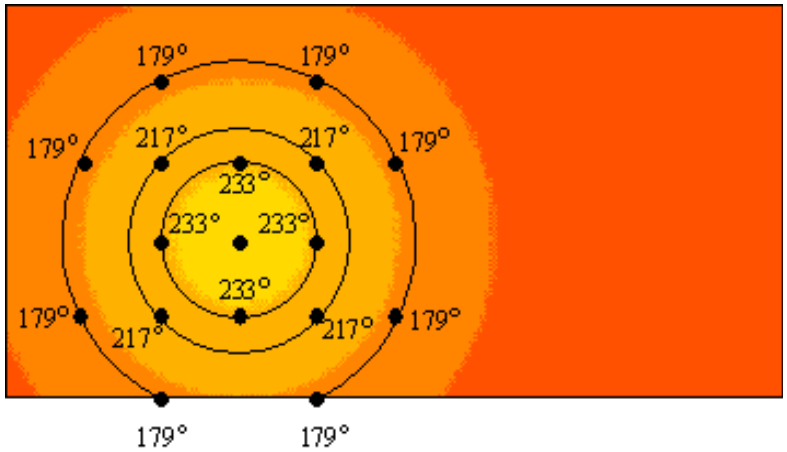
$$g(x,y) = 70 + 180e^{-(x-3)^2/10-(y-2)^2/10} \tag{6}$$

which for integer pairs (x,y) *in* $[0,10] \times [0,5]$ *is given by*



(7)

and which has the isotherms



(8)

33. Use (6) to determine the level curves with levels $k = 179°F$, $217°F$, $233°F$, and $250°F$. Locate the point $(6, 3)$ on a level curve and calculate the gradient at that point. What is significant about the gradient at that point? Which direction is it pointing in?

34. What is the rate of change of $g(x, y)$ in (2) at $(1, 1)$ in the direction of

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Why would we expect it to be less than $48.82°F$ *per inch*?

35. In what direction is the temperature field

$$g(x, y) = 70 + 180e^{-(x-3)^2/10-(y-2)^2/10}$$

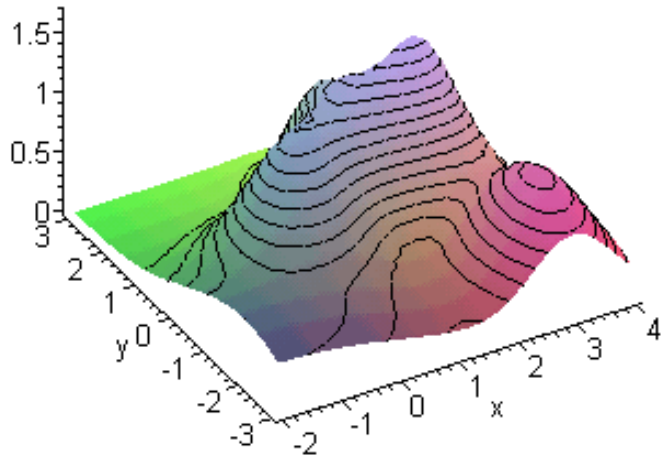
increasing the fastest at the point $(1, 0)$? At the point $(0, 2)$? Compare your result with the temperatures in (7).

36. Show that the gradient of the temperature function

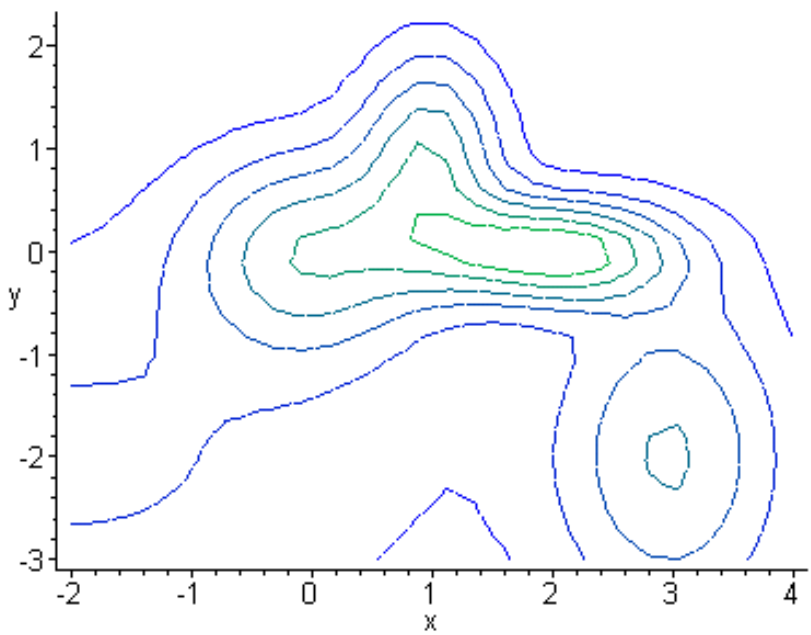
$$g(x, y) = 70 + 180e^{-(x-3)^2/10-(y-2)^2/10}$$

always points toward $(3, 2)$. What is the significance of this result?

37. Consider the "mountain" shown below, where units are in thousands of feet:



If the family of curves below corresponds to



sketch the path of least resistance from the highest point on the mountain to its base.

38. Show that if $F(x,y) = y - f(x)$ and that if $f(x)$ is differentiable everywhere, then the curve

$$\mathbf{r}(t) = \langle t,\ f(t) + k,\ k \rangle,\ t\ in\ (-\infty, \infty)$$

is a level curve of $z = F(x, y)$ with level k. What is $\mathbf{r}'(t) \cdot \nabla F$ when $x = t$ and $y = f(t) + k$. Explain.

39. Write to Learn: Show that the curve $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t) \rangle$ is a parametrization of $f(x, y) = x^2 + y^2$ with level 16. Then find $\mathbf{T}$ and $\mathbf{N}$ at $t = \pi/6$. How is $\mathbf{N}$ at $t = \pi/6$ related to ∇f at the point $\left(2, 2\sqrt{3}\right)$? Write a short essay explaining this relationship.

40. *Write to Learn: Obtain a city map and fix a street corner on that map to be the origin. Define a function $f(x, y)$ to be the driving distance from that street corner to a point (x, y), where the driving distance is the shortest possible distance from the origin to (x, y) along the roads on the map. What are the level curves of the driving distance function? Sketch a few of them. What is the path of least resistance at a given point? Write an essay describing the $f(x, y)$ function, its level curves, and its paths of least resistance.

In exercises 41 - 44, we consider the vectors

$$\begin{aligned} \mathbf{v}_{inc} &= \left\langle \frac{f_x(\mathbf{p})}{\|\nabla f(\mathbf{p})\|}, \frac{f_y(\mathbf{p})}{\|\nabla f(\mathbf{p})\|}, \|\nabla f(\mathbf{p})\| \right\rangle, \quad \mathbf{v}_{level} = \left\langle \frac{-f_y(\mathbf{p})}{\|\nabla f(\mathbf{p})\|}, \frac{f_x(\mathbf{p})}{\|\nabla f(\mathbf{p})\|}, 0 \right\rangle \\ \mathbf{v}_{perp} &= \langle -f_x(\mathbf{p}), -f_y(\mathbf{p}), 1 \rangle \end{aligned}$$

at a point $p = (p, q)$ with the assumption that $\nabla f(\mathbf{p})$ exists and is non-zero.

41. Show that $\mathbf{v}_{level} \perp \mathbf{v}_{perp}$ and that $\mathbf{v}_{level}$ is a unit vector. What is $(D_{\mathbf{u}} f)(\mathbf{p})$ when $\mathbf{u} = \mathbf{v}_{level}$? What is significant about these results about $\mathbf{v}_{level}$?

42. Explain why $\mathbf{v}_{level}$ and $\mathbf{v}_{perp}$ are tangent to the surface $z = f(x, y)$ at the point $(p, q, f(\mathbf{p}))$. Then show that

$$\mathbf{v}_{perp} = \mathbf{v}_{inc} \times \mathbf{v}_{level}$$

What is the significance of this result?

43. What is the equation of the plane through $(p, q, f(\mathbf{p}))$ with normal $\mathbf{v}_{perp}$? What is the significance of this result?

44. Write to Learn: In a short essay, explain why if $\mathbf{r}(t)$ is a curve on the surface $z = f(x, y)$ and if

$$\mathbf{r}(0) = (p, q, f(\mathbf{p}))$$

then there are numbers a and b such that

$$\mathbf{r}'(0) = a\mathbf{v}_{inc} + b\mathbf{v}_{level}$$