# The Chain Rule

**The Chain Rule**

In this section, we generalize the *chain rule* to functions of more than one variable. In particular, we will show that the product in the single-variable chain rule extends to an *inner product* in the general case.

To begin with, suppose that $C$ is a curve in the $xy$-plane with a smooth parametrization $\mathbf{x}(t) = \langle x(t), y(t) \rangle$, $t$ in $[a, b]$, and suppose that $f(\mathbf{x})$ is differentiable at each point on $C$. In the last part of the previous section, we saw that Definition 5.1b implies that on some neighborhood of 0, there is a continuous function $\varepsilon(\Delta\mathbf{x})$ with $\varepsilon(0) = 0$ such that

$$|\Delta f - \nabla f \cdot \Delta\mathbf{x}| < \varepsilon(\boldsymbol{\Delta}\mathbf{x}) \|\Delta\mathbf{x}\|$$

If we now suppose that $\Delta\mathbf{x} = \mathbf{x}(t + \Delta t) - \mathbf{x}(t)$ for some small nonzero change $\Delta t$ in the parameter, then

$$\left| \frac{\Delta f}{\Delta t} - \nabla f \cdot \frac{\Delta\mathbf{x}}{\Delta t} \right| < \varepsilon(\boldsymbol{\Delta}\mathbf{x}) \left\| \frac{\Delta\mathbf{x}}{\Delta t} \right\|$$

If $\Delta t$ approaches 0, then $\Delta x$ also approaches 0 and so

$$\lim_{\Delta t \to 0} \left| \frac{\Delta f}{\Delta t} - \nabla f \cdot \frac{\Delta\mathbf{x}}{\Delta t} \right| \le \lim_{\Delta t \to 0} \left( \varepsilon(\boldsymbol{\Delta}\mathbf{x}) \left\| \frac{\Delta\mathbf{x}}{\Delta t} \right\| \right)$$

which leads to

$$\left| \frac{df}{dt} - \nabla f \cdot \frac{d\mathbf{x}}{dt} \right| \le \varepsilon(0) \left\| \frac{d\mathbf{x}}{dt} \right\|$$

Since $\varepsilon(0) = 0$, this implies that following:

> **The Chain Rule**: If $\mathbf{x}(t)$ is differentiable at each point $t$ in an interval $[a, b]$, and if $f(\mathbf{x})$ is differentiable at each $\mathbf{x}(t)$, $t$ in $[a, b]$, then
>
> $$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{x}}{dt}$$

Alternatively, the chain rule can be written as

$$\frac{d}{dt} f(\mathbf{r}) = \nabla f \cdot \mathbf{v} \tag{1}$$

where $\mathbf{v} = \mathbf{x}'(t)$ is the velocity vector of $\mathbf{x}(t)$. Also, if we let $df/d\mathbf{x}$ denote the gradient $\nabla f$ (i.e., $df/d\mathbf{x} = \nabla f$ ) then the chain rule can be writtten in the form

$$\frac{df}{dt} = \frac{df}{d\mathbf{x}} \cdot \frac{d\mathbf{x}}{dt}$$

which is reminiscent of the chain rule for functions of a single variable.

*EXAMPLE 1* Evaluate $df/dt$ using (1) given that $f(x,y) = x^2 - y^2$ and $\mathbf{x}(t) = \langle \sin(t), \cos(t) \rangle$

**Solution:** Since $\nabla f = \langle 2x, -2y \rangle$ and

$$\mathbf{v} = \mathbf{x}'(t) = \langle \cos(t), -\sin(t) \rangle$$

the chain rule (1) implies that

$$\begin{aligned} \frac{df}{dt} &= \nabla f \cdot \mathbf{v} \\ &= \langle 2x, -2y \rangle \cdot \langle \cos(t), -\sin(t) \rangle \\ &= 2x\cos(t) + 2y\sin(t) \end{aligned}$$

However, $\mathbf{x}(t) = \langle \sin(t), \cos(t) \rangle$ implies that $x = \sin(t)$ and $y = \cos(t)$, so that

$$\begin{aligned} \frac{df}{dt} &= 2\sin(t)\cos(t) + 2\cos(t)\sin(t) \\ &= 4\sin(t)\cos(t) \\ &= 2\sin(2t) \end{aligned}$$

Similarly, if $w = U(x,y,z)$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve in $R^3$, then we let $w = U(\mathbf{r})$ and let

$$\nabla U = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle$$

be the *gradient* of $U$, so that the chain rule for 3 variables can be written

$$\frac{dU}{dt} = \nabla U \cdot \frac{d\mathbf{r}}{dt}$$

*EXAMPLE 2* Evaluate $dU/dt$ using (1) given that

$$U(x,y,z) = xy + z^2$$

and $\mathbf{r}(t) = \left\langle e^t, e^{-t}, t^3 \right\rangle$.

**Solution:** Since $\nabla U = \langle y, x, 2z \rangle$ and $\mathbf{r}' = \left\langle e^t, -e^{-t}, 3t^2 \right\rangle$, the chain rule says that

$$\begin{aligned} \frac{dU}{dt} &= \nabla U \cdot \frac{d\mathbf{r}}{dt} \\ &= \langle y, x, 2z \rangle \cdot \left\langle e^t, -e^{-t}, 3t^2 \right\rangle \\ &= ye^t - xe^{-t} + 6zt^2 \end{aligned}$$

Since $x = e^t$, $y = e^{-t}$, and $z = t^3$, we have

$$\frac{dU}{dt} = e^{-t}e^t - e^t e^{-t} + 6t^3 t^2 = 6t^5$$

**Check your Reading:** How many trig identities did we use in example 1?

## The Chain Rule in Non-Vector Form

Since $\nabla f = \langle f_x, f_y \rangle$, the expression $\nabla f \cdot \mathbf{v}$ can be written as

$$\nabla f \cdot \mathbf{v} = \langle f_x, f_y \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Thus, the chain rule can also be written as follows:

**The Chain Rule(non-vector form):** Suppose that $x(t)$ and $y(t)$ are differentiable at $t_0$ and that $f(x, y)$ is differentiable at $(x(t_0), y(t_0))$. If $w = f(x, y)$, then $w(t)$ is differentiable at $t_0$ and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

where $dw/dt = w'(t_0)$.

Equivalently, the chain rule produces the same derivative $dw/dt$ that we would obtain directly by substituting for $x$ and $y$ in $f(x, y)$ and differentiating with respect to $t$.

*EXAMPLE 3* Find $dw/dt$ given that $w = x^2 + y^3$ and that $x = t^3$, $y = t^5$.

**Solution:** The first partial derivatives of $w = x^2 + y^3$ are

$$\frac{\partial w}{\partial x} = 2x, \qquad \frac{\partial w}{\partial y} = 3y^2$$

As a result, the chain rule says that

$$\begin{aligned}
\frac{dw}{dt} &= 2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} \\
&= 2x\left(3t^2\right) + 3y^2\left(5t^4\right) \\
&= 2t^3\left(3t^2\right) + 3t^{10}\left(5t^4\right) \\
&= 6t^5 + 15t^{14}
\end{aligned}$$

Notice that we would have obtained the same result if we had substituted

$$w = \left(t^3\right)^2 + \left(t^5\right)^3 = t^6 + t^{15}$$

and then calculated $dw/dt$.

Likewise, if $w = f(x, y, z)$ where $x$, $y$, and $z$ are functions of $t$, then

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

if $x$ and $y$ are differentiable at $t_0$ and $f(x, y)$ is differentiable at $(x(t_0), y(t_0))$.

*EXAMPLE 4* Find $dw/dt$ given that $w = \cos(xy) + z$ and that $x = \pi e^t$, $y = e^{-t}$, and $z = t^2$

**Solution:** The first partial derivatives are

$$\frac{\partial w}{\partial x} = -y\sin(xy), \quad \frac{\partial w}{\partial y} = -x\sin(xy), \quad \frac{\partial w}{\partial z} = 1$$

As a result, the chain rule says that

$$\frac{dw}{dt} = -y\sin(xy)\frac{dx}{dt} - x\sin(xy)\frac{dy}{dt} + \frac{dz}{dt}$$

and since $dx/dt = \pi e^t$ , $dy/dt = -e^{-t}$, and $dz/dt = 2t$, we have

$$\begin{aligned} \frac{dw}{dt} &= -e^{-t}\sin\left(\pi e^t e^{-t}\right)\left(\pi e^t\right) - \pi e^t \sin\left(\pi e^t e^{-t}\right)\left(-e^{-t}\right) + 2t \\ &= -\pi e^t e^{-t}\sin\left(\pi e^t e^{-t}\right) + \pi e^t e^{-t}\sin\left(\pi e^t e^{-t}\right) + 2t \\ &= -\pi\sin(\pi) + \pi\sin(\pi) + 2t \\ &= 2t \end{aligned}$$

**Check your Reading:** Substitute $x = \pi e^t$ , $y = e^{-t}$, and $z = t^2$ into $w = \cos(xy) + z$ to reveal another reason why we should have $dw/dt = 2t$ in example 4.

## The Chain Rule for Partial Derivatives

If $w = f(x, y)$ and if $x$ and $y$ are functions of variables $u$ and $v$, then the chain rule yields

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \qquad \textit{and} \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

That is, the chain rule for partial derivatives is a natural extension of the chain rule for ordinary derivatives.

*EXAMPLE 5* Find $\partial w/\partial u$ and $\partial w/\partial v$ when $w = x^2 + xy$ and $x = u^2 v$, $y = uv^2$.

**Solution:** To begin with, the first partial derivatives of $w = x^2 + xy$ are

$$\frac{\partial w}{\partial x} = 2x + y, \quad \frac{\partial w}{\partial y} = x,$$

while the partial derivatives of $x$ and $y$ with respect to $u$ are

$$\frac{\partial x}{\partial u} = 2uv, \quad \frac{\partial y}{\partial u} = v^2$$

As a result, the chain rule says that

$$\frac{\partial w}{\partial u} = (2x + y)\,\frac{\partial x}{\partial u} + x\,\frac{\partial y}{\partial u}$$

Substitution for $x$, $y$ and their derivatives yields

$$\begin{aligned} \frac{\partial w}{\partial u} &= \left(2u^2 v + uv^2\right) 2uv + u^2 v \left(v^2\right) \\ &= 4u^3 v^2 + 3u^2 v^3 \end{aligned}$$

To evaluate $\partial w/\partial v$, we substitute the first partial derivatives to obtain

$$\frac{\partial w}{\partial v} = (2x + y)\,\frac{\partial x}{\partial v} + x\,\frac{\partial y}{\partial v}$$

The partial derivatives of $x$ and $y$ with respect to $v$ are

$$\frac{\partial x}{\partial v} = u^2, \quad \frac{\partial y}{\partial v} = 2uv$$

so that substitution for $x$, $y$ and their derivatives yields

$$\begin{aligned} \frac{\partial w}{\partial v} &= \left(2u^2 v + uv^2\right) u^2 + u^2 v\,(2uv) \\ &= 2u^4 v + 3u^3 v^2 \end{aligned}$$

**Check your Reading:** Explain why $w = u^4 v^2 + u^3 v^3$

**Applications**

The chain rule is often used to prove theorems and develop new techniques. For example, if a curve $g(x,y)=k$ implicitly defines $y$ as a function of $x$, then $x=t$, $y=f(t)$ for some unknown function $f(t)$. Thus,

$$\begin{aligned} \frac{d}{dt}g(x,y) &= \frac{d}{dt}k \\ g_x\frac{dx}{dt}+g_y\frac{dy}{dt} &= 0 \\ g_x+g_y\frac{df}{dt} &= 0 \end{aligned}$$

If we let $y'=df/dt$, then $g_y y'=-g_x$ and

$$y'=\frac{-g_x}{g_y}$$

That is, *implicit differentiation* is an application of the chain rule.

*EXAMPLE 6* Find $y'$ if $y$ is implicitly defined to be a function of $x$ and

$$x^2+y^2=1$$

**Solution:** Let $g(x,y)=x^2+y^2$. Then $g_x=2x$ and $g_y=2y$ and

$$y'=\frac{-g_x}{g_y}=\frac{-2x}{2y}=\frac{-x}{y}$$

In many applications, we are asked to consider functions of the form

$$w(t)=\int_a^{p(t)} g(u,q(t))\,du$$

To find $dw/dt$, we often let $x=p(t)$ and $y=q(t)$ and then apply the chain rule to

$$w=\int_a^x g(u,y)\,du$$

To find $w_y$, we will assume that the derivative with respect to $y$ can be moved into the integrand (see exercise 35).

*EXAMPLE 7* Find $dw/dt$ given

$$w=\int_0^t \sin\left(u^2+t^2\right)du$$

**Solution:** To begin with, we define

$$w = \int_0^x \sin\left(u^2 + y\right) du$$

where $x = t$ and $y = t^2$. The first partial with respect to $x$ is

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \int_0^x \sin\left(u^2 + y\right) du = \sin\left(x^2 + y\right)$$

Differentiating under the integral yields

$$\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial}{\partial y} \int_0^x \sin\left(u^2 + y\right) du \\ &= \int_0^x \frac{\partial}{\partial y} \sin\left(u^2 + y\right) du \\ &= \int_0^x \cos\left(u^2 + y\right) du \end{aligned}$$

Since $x'(t) = 1$ and $y'(t) = 2t$, the chain rule then yields

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= \sin\left(x^2 + y\right)\frac{dx}{dt} + \left(\int_0^x \cos\left(u^2 + y\right) du\right)\frac{dy}{dt} \end{aligned}$$

Substituting $x = t$, $y = t^2$, $x' = 1$, and $y' = 2t$ yields

$$w'(t) = \sin\left(2t^2\right) + 2t \int_0^t \cos\left(u^2 + t^2\right) du$$

## Exercises

*Find the specified derivative(s) using the chain rule:*

1. $\frac{dw}{dt}$ if $w = x^2y^2$ and $x = t^4$, $y = t^5$
2. $\frac{dw}{dt}$ if $w = x^3y$ and $x = t$, $y = t^2$
3. $\frac{dw}{dt}$ if $w = x^2 + y^2$ and $x = \cos(t)$, $y = \sin(t)$
4. $\frac{dw}{dt}$ if $w = x^2 - y^2$ and $x = \cosh(t)$, $y = \sinh(t)$
5. $\frac{dw}{dt}$ if $w = xyz$ and $x = e^t$, $y = e^{-t}$, $z = t^5$
6. $\frac{dw}{dt}$ if $w = x^2 - zy^2$ and $x = \sec(t)$, $y = \tan(t)$, $z = 1$
7. $\frac{dw}{dt}$ if $w = x^2 + 2xy + z^3$ and $x = \cos(t)$, $y = \sin(t)$, $z = t$
8. $\frac{dw}{dt}$ if $w = x^3y^2z$ and $x = e^t$, $y = e^{-t}$, $z = e^{-t}$
9. $\frac{dw}{dt}$ if $w = \tan^{-1}\left(\frac{y}{x}\right)$ and $x = \sin(t)$, $y = \cos(t)$
10. $\frac{dw}{dt}$ if $w = \sin^{-1}(xy)$ and $x = \cos(t)$, $y = \tan(t)$

11. $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ if $w = x^2 + y^2$ and $x = u^2 v$, $y = (u+v)^3$

12. $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ if $w = x \sin(y)$ and $x = 2uv$, $y = u^2 - v^2$

13. $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ if $w = \sin(x) \cot(y)$ $x = \sin^{-1}(uv)$, $y = \tan^{-1}(uv)$

14. $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ if $w = \sin(xy) e^{xy}$ $x = uv^{-1} \ln(v)$, $y = u^{-1} v$

15. $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}$ if $w = u^2 + v^2$ and $u = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$, $v = \frac{1}{t} e^{-x^2/(2t)}$

16. $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ if $w = u^2 + v^2$ and $u = \sin(x - ct)$, $v = \cos(x - ct)$

17. $F'(t)$ if $F(t) = \int_0^t \sin\left(u^2 t\right) du$

18. $F'(t)$ if $F(t) = \int_0^t e^{-u^2/t} du$

19. $F'(t)$ if $F(t) = \int_t^{t+h} \sin(u - t)\, dt$

20. $F'(t)$ if $F(t) = \int_0^t \frac{e^{-u}}{u^2 + t^2} du$

*Compute $\nabla f$ and then use it to compute $df/dt$ using the vector form of the chain rule.*

21. $f(x, y) = x^2 + y^3$
$\mathbf{r}(t) = \left\langle t^2, t^3 \right\rangle$

22. $f(x, y) = x^3 y^2$
$\mathbf{r}(t) = \left\langle t^2, t^3 \right\rangle$

23. $f(x, y) = x^2 + 2xy + z^2$
$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

24. $f(x, y) = x^3 y^2 z$
$\mathbf{r}(t) = \langle e^t, e^{-t}, e^{-t} \rangle$

25. $f(x, y) = xz \cos(y)$
$\mathbf{r}(t) = \langle \sin(t), t, \csc(2t) \rangle$

26. $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + z$
$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$

**27.** Find $y'$ given that $y$ is implicitly defined as a function of $x$ by

$$x^2 + y^2 = 2xy + 1$$

**28.** Find $y'$ given that $y$ is implicitly defined as a function of $x$ by

$$x \sin(xy) = y^2$$

**29.** Compute $dw/dt$ for $w = x^2 - y^2$, $x = \cos(t)$, $y = \sin(t)$ in two different ways:

1. (a) By substituting $x = \cos(t)$, $y = \sin(t)$ into $w = x^2 - y^2$, simplifying, and computing the derivative.
   (b) By using the chain rule for two variables, and then simplifying.

**30.** Compute $dw/dt$ for $w = x^3 + xy^2$, $x = \cos(t)$, $y = \sin(t)$ in two different ways:

1. (a) By substituting $x = \cos(t)$, $y = \sin(t)$ into $w = x^3 + xy^2$, simplifying, and computing the derivative.
   (b) By using the chain rule for two variables, and then simplifying.

**31.** Prove that the derivative of a sum is the sum of the derivatives by applying the chain rule for 2 variables to

$$w = x + y$$

where $x = f(t)$ and $y = g(t)$.

**32.** Prove that if $w = f(\mathbf{r})$ where $\mathbf{r} = \langle g(u,v), h(u,v) \rangle$, then

$$\frac{\partial w}{\partial u} = \frac{df}{d\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial u}, \qquad \frac{\partial w}{\partial v} = \frac{df}{d\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial v},$$

**33.** Prove the *product rule* by applying the chain rule for 2 variables to

$$w = xy$$

where $x = f(t)$ and $y = g(t)$.
**34.** Prove the *quotient rule* by applying the chain rule for 2 variables to

$$w = \frac{x}{y}$$

where $x = f(t)$ and $y = g(t)$.
**35.** Suppose that $K(x,u)$ is differentiable in $x$ and suppose that for all $\varepsilon > 0$, there is a interval $(p,q)$ such that if $x$ is in $(p,q)$, then

$$\left| \frac{K(x+h,u) - K(x,u)}{h} - K_x(x,u) \right| < \varepsilon \qquad (2)$$

independent of $u$. Show that

$$\frac{d}{dx} \int_a^b K(x,u)\, du = \int_a^b \frac{\partial K}{\partial x}(x,u)\, du$$

1. (a) Let $f(x) = \int_a^b K(x,u)\, du$ and show that

$$\frac{f(x+h) - f(x)}{h} = \int_a^b \left( \frac{K(x+h,u) - K(x,u)}{h} \right) du$$

   (b) Show that if $x$ is in $(p,q)$ where $(p,q)$ is an interval on which (2) holds, then

$$\left| \frac{f(x+h) - f(x)}{h} - \int_a^b \frac{\partial K}{\partial x}(x,u)\, du \right| < \varepsilon (b-a)$$

   What does this imply about $f'(x)$?

**36.** This exercise uses 2 different methods to differentiate the indefinite integral

$$F(t) = \int_0^t e^{u-t} du$$

1. (a) Find $F'(t)$ using the chain rule for functions of 2 variables.

   (b) Write $F(t)$ as the product of two functions of $t$ and apply the product rule. Is the result the same in both cases?

**37.** Show that the convolution function

$$y(t) = \int_0^t e^{t-u} f(u)\, du$$

is a solution to $y' - y = f(t)$.

**38.** Show that the convolution function

$$y(t) = \int_0^t \sin(t-u) f(u)\, du$$

is a solution to $y'' + y = f(t)$.

**39.** If $N(t)$ is the population of a certain bacteria colony at time $t$, then

$$N(t) = \begin{array}{c}\text{proportion of the}\\ \text{initial population}\\ \text{that survives to time } t\end{array} + \begin{array}{c}\text{the accumulation of}\\ \text{those born in } [0,t] \text{ who}\\ \text{who survive to time } t\end{array}$$

If $P(t)$ is the probability that an individual born at time 0 will survive to age $t$, then $N_0 P(t)$ is the proportion who survive to time $t$. If $b$ is the intrinsic birth rate, then $bN(t)\Delta t$ is the number of births from time $t$ to time $t + \Delta t$ for $\Delta t$ small.

a. Explain why $P(t-\tau)$ is the probability that an individual born at time $\tau$ will survive to time $t$, and use this to explain why the population can be modeled by

$$N(t) = P(t) N_0 + \int_0^t bN(\tau) P(t-\tau)\, d\tau \tag{3}$$

b. Differentiate both sides of (3) to find an equation for $N'(t)$.

c. Show that if $P(t) = e^{-\gamma t}$ where $\gamma > 0$ is constant, then what separable differential equation does $b$ reduce to?

**40.** Repeat exercise 39 given that the number of births from time $t$ to time $t + \Delta t$ is

$$bN(t)(K - N(t))$$

where $K$ is a constant known as the *carrying capacity* for the population.

**41.** Show that if $f = f(g,h)$ where $g = g(x,y)$ and $h = h(x,y)$, then

$$\nabla f = f_x \nabla g + f_y \nabla h$$

**40.** Show that if $z = f(x(t), y(t))$, then

$$\begin{aligned}\frac{d^2 z}{dt^2} &= f_{xx}\left(\frac{dx}{dt}\right)^2 + 2f_{xy}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + f_{yy}\left(\frac{dy}{dt}\right)^2 \\ &\quad + f_x \frac{d^2 x}{dt^2} + f_y \frac{d^2 y}{dt^2}\end{aligned}$$

**43.** Suppose that $z = f(x,y)$ and that $x = p + mt$ and $y = q + nt$, where $m$, $n$, $p$, and $q$ are constants. Show that

$$z'(t) = mf_x + nf_y \qquad \textit{and} \qquad z''(t) = f_{xx}m^2 + 2mnf_{xy} + f_{yy}n^2$$

**44.** A function $f(x,y)$ is said to be *homogeneous of degree* $n$ if

$$f(tx,ty) = t^n f(x,y) \tag{4}$$

for all real numbers $t$. For example, $f(x,y) = x^3 + 3xy^2$ is homogeneous of degree 3 since

$$\begin{aligned} f(tx,ty) &= (tx)^3 + 3(tx)(ty)^2 \\ &= t^3x^2 + t^3 3xy^2 \\ &= t^3 f(x,y) \end{aligned}$$

Show that if a differentiable function $f(x,y)$ is homogeneous of degree $n$, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$$

(hint: Differentiate both sides of (4) with respect to $t$—use the chain rule to differentiate $f(tx,ty)$—and then let $t = 1$).