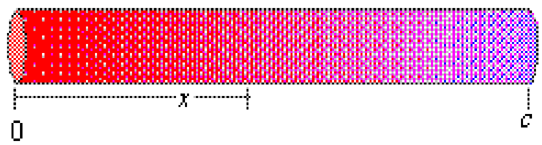
Partial Differential Equations

Partial Differential Equations

Much of modern science, engineering, and mathematics is based on the study of partial differential equations, where a *partial differential equation* is an equation involving partial derivatives which implicitly defines a function of 2 or more variables.

For example, if $u(x,t)$ is the temperature of a metal bar at a distance x from the initial end of the bar,



then under suitable conditions $u(x,t)$ is a solution to the *heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where k is a constant. As another example, consider that if $u(x,t)$ is the displacement of a string a time t, then the vibration of the string is likely to satisfy the *one dimensional wave equation* for a constant, which is

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

When a partial differential equation occurs in an application, our goal is usually that of solving the equation, where a given function is a *solution* of a partial differential equation if it is implicitly defined by that equation. That is, a solution is a function that satisfies the equation.

EXAMPLE 1 Show that if a is a constant, then $u(x,y) = \sin(at)\cos(x)$ is a solution to

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \tag{2}$$

Solution: Since a is constant, the partials with respect to t are

$$\frac{\partial u}{\partial t} = a\cos(at)\cos(x), \qquad \frac{\partial^2 u}{\partial t^2} = -a^2\sin(at)\sin(x) \tag{3}$$

Moreover, $u_x = -\sin(at)\sin(x)$ and $u_{xx} = -\sin(at)\cos(x)$, so that

$$a^2\frac{\partial^2 u}{\partial x^2} = -a^2\sin(at)\cos(x) \tag{4}$$

Since (3) and (4) are the same, $u(x,t) = \sin(at)\cos(x)$ is a solution to (2).

EXAMPLE 2 Show that $u(x,t) = e^y \sin(x)$ is a solution to *Laplace's Equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solution: To begin with, $u_x = e^y \cos(x)$ and $u_{xx} = -e^y \sin(x)$. Moreover, $u_y = e^y \sin(x)$ and $u_{yy} = e^y \sin(x)$, so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^y \sin(x) + e^y \sin(x) = 0$$

Check your Reading: Why are u, u_y, and u_{yy} the same as u in example 2?

Separation of Variables

Solutions to many (but not all!) partial differential equations can be obtained using the technique known as *separation of variables.* It is based on the fact that if $f(x)$ and $g(t)$ are functions of independent variables x, t respectively and if

$$f(x) = g(t)$$

then there must be a constant λ for which $f(x) = \lambda$ and $g(t) = \lambda$. (The proof is straightforward, in that

$$\begin{aligned} \frac{\partial}{\partial x} f(x) &= \frac{\partial}{\partial x} g(t) = 0 \quad \Longrightarrow \quad f'(x) = 0 \Longrightarrow f(x) \ \textit{constant} \\ \frac{\partial}{\partial t} g(t) &= \frac{\partial}{\partial t} f(x) = 0 \quad \Longrightarrow \quad g'(t) = 0 \Longrightarrow g(x) \ \textit{constant}) \end{aligned}$$

In separation of variables, we first assume that the solution is of the *separated form*

$$u(x,t) = X(x)\,T(t)$$

We then substitute the separated form into the equation, and *if possible*, move the x-terms to one side and the t-terms to the other. If not possible, then this method will not work; and correspondingly, we say that the partial differential equation is not separable.

Once separated, the two sides of the equation must be constant, thus requiring the solutions to ordinary differential equations. A table of solutions to common differential equations is given below:

Equation	General Solution
$y'' + \omega^2 y = 0$	$y(x) = A\cos(\omega x) + B\sin(\omega x)$
$y' = ky$	$y(t) = Pe^{kt}$
$y'' - \omega^2 y = 0$	$y(x) = A\cosh(\omega x) + B\sinh(\omega x)$

The product of $X(x)$ and $T(t)$ is the *separated solution* of the partial differential equation.

EXAMPLE 3 For k constant, find the separated solution to the *Heat Equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Solution: To do so, we substitute $u(x,t) = X(x)T(t)$ into the equation to obtain

$$\frac{\partial}{\partial t}(X(x)T(t)) = k\frac{\partial^2}{\partial x^2}(X(x)T(t))$$

Since $X(x)$ does not depend on t, and since $T(t)$ does not depend on x, we obtain

$$X(x)\frac{\partial}{\partial t}T(t) = kT(t)\frac{\partial^2}{\partial x^2}X(x)$$

which after evaluating the derivatives simplifies to

$$X(x)T'(t) = kT(t)X''(x)$$

To separate the variables, we divide throughout by $kX(x)T(t)$:

$$\frac{X(x)T'(t)}{kX(x)T(t)} = \frac{kT(t)X''(x)}{kX(x)T(t)}$$

This in turn simplifies to

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

Thus, there is a constant λ such that

$$\frac{T'}{kT} = \lambda \qquad \text{and} \qquad \frac{X''}{X} = \lambda$$

These in turn reduce to the differential equations

$$T' = \lambda kT \qquad \text{and} \qquad X'' = \lambda X$$

The solution to the first is an exponential function of the form

$$T(t) = Pe^{\lambda kt}$$

If $\lambda > 0$, however, then temperature would grow to ∞, which is not physically possible. Thus, we assume that λ is negative, which is to say that $\lambda = -\omega^2$ for some number ω. As a result, we have

$$X'' = -\omega^2 X \qquad \text{or} \qquad X'' + \omega^2 X = 0$$

The equation $X'' + \omega^2 X = 0$ is a harmonic oscillator, which has a solution

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Consequently, the *separated solution* for the heat equation is

$$u(x,t) = X(x)\,T(t) = Pe^{-\omega^2 kt}\left(A\cos(\omega x) + B\sin(\omega x)\right)$$

It is important to note that in general a **separated solution** to a partial differential equation is **not the only solution** or form of a solution. Indeed, in the exercises, we will show that

$$u(x,t) = \frac{1}{\sqrt{kt}}e^{-x^2/(4kt)}$$

is also a solution to the heat equation in example 3.

As a simpler example, consider that $F(x,y) = y - x^2$ is a solution to the partial differential equation

$$F_x + 2xF_y = 0$$

This is because substituting $F_x = -2x$ and $F_y = 1$ into the equation yields

$$F_x + 2xF_y = -2x + 2x \cdot 1 = 0$$

Now let's obtain a different solution by assuming a separated solution of the form $F(x,y) = X(x)\,Y(y)$.

EXAMPLE 4 Find the separated solution to $F_x + 2xF_y = 0$.

Solution: The separated form $F(x,y) = X(x)\,Y(y)$ results in

$$\frac{\partial}{\partial x}\left(X(x)\,Y(y)\right) + 2x\frac{\partial}{\partial y}\left(X(x)\,Y(y)\right) = 0$$

which in turn leads to

$$X'(x)\,Y(y) = -2xX(x)\,Y'(y)$$

Dividing both sides by $X(x)\,Y(y)$ leads to

$$\frac{X'(x)}{-2xX(x)} = \frac{Y'(y)}{Y(y)}$$

However, a function of x can be equal to a function of y for all x and y only if both functions are constant. Thus, there is a constant λ such that

$$\frac{X'(x)}{-2xX(x)} = \lambda \qquad \text{and} \qquad \frac{Y'(y)}{Y(y)} = \lambda$$

It follows that $Y'(y) = \lambda Y(y)$, which implies that $Y(y) = C_1 e^{\lambda y}$. However, $X'(x) = \lambda x X(x)$, so that separation of variables yields

$$\frac{dX}{dx} = -2\lambda x X \qquad \Longrightarrow \qquad \frac{dX}{X} = \lambda x dx$$

Thus, $\int dX/X = \lambda \int x dx$, which yields

$$\begin{aligned} \ln|X| &= -\lambda x^2 + C_2 \\ |X| &= e^{-\lambda x^2 + C_2} \\ X(x) &= \pm e^{C_2} e^{-\lambda x^2} \end{aligned}$$

Thus, if we let $C_3 = \pm e^{C_2}$, then $Y(y) = C_3 \exp\left(x^2/2\right)$ and the separated solution is

$$F(x,y) = Ce^{-\lambda x^2} e^{\lambda y} = Ce^{\lambda\left(y - x^2\right)}$$

where $C = C_1 C_3$ is an arbitrary constant.

Notice that there are similarities between the separated solution

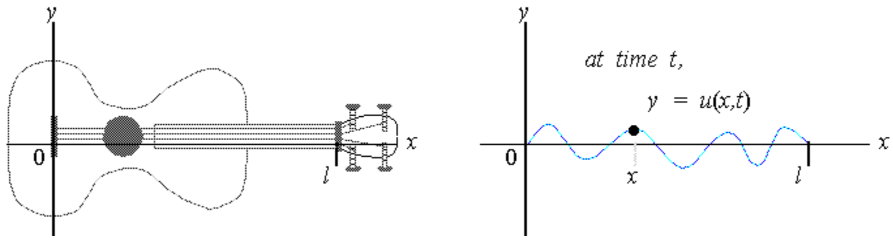
$$F(x,y) = Ce^{\lambda\left(y-x^2\right)}$$

and the other solution we stated earlier, $F(x,y) = y - x^2$. However, the two solutions are clearly not the same.

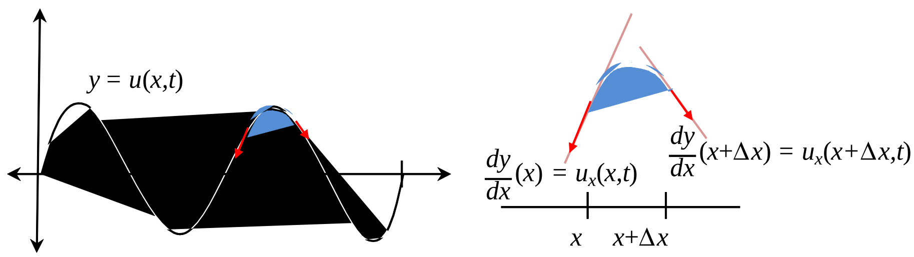
Check your Reading: Why is this method called *separation of variables?*

Boundary Conditions

Partial differential equations often occur with *boundary conditions,* which are constraints on the solution at different points in space. To illustrate how boundary conditions arise in applications, let us suppose that $u(x,t)$ is the displacement at x in $[0,l]$ of a string of length l at time t:



Tension on a short section of the string over the interval $[x, x+\Delta x]$ is along the tangents to the endpoints,



Thus, the net tension responsible for pulling the string toward the x-axis is proportional to the difference in the slopes,

$$Net\ Tension = k\left(\ u_x\left(x+\Delta x, t\right) - u_x\left(x, t\right)\ \right)$$

where k is the tension constant (see http://en.wikipedia.org/wiki/Vibrating_string#Derivation for details). Consequently, if μ is the mass-density of the string (mass per unit length), then mass times acceleration equal to the force of tension yields

$$\mu \Delta x \frac{\partial^2 u}{\partial t^2} = k\left(\ u_x\left(x+\Delta x, t\right) - u_x\left(x, t\right)\ \right)$$

for arbitrarily small Δx. Solving for u_{tt} and letting Δx approach 0 yields

$$\frac{\partial^2 u}{\partial t^2} = \frac{k}{\mu} \lim_{\Delta x \to 0} \frac{u_x\left(x+\Delta x, t\right) - u_x\left(x, t\right)}{\Delta x} = \frac{k}{\mu} \frac{\partial^2 u}{\partial x^2}$$

so that if we let $a^2 = k/\mu$, then the partial differential equation describing the motion of the string is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{5}$$

which is the one-dimensional wave equation.

Moreover, since the string is fixed at $x = 0$ and $x = l$, we also have the *boundary conditions*

$$u\left(0, t\right) = 0 \qquad and \qquad u\left(l, t\right) = 0 \tag{6}$$

for all times t. If we *avoid the trivial solution* (that of no vibration, $u = 0$), then these boundary conditions can be used to determine some of the arbitrary constants in the separated solution.

> *EXAMPLE 5* Find the solution of the one dimensional wave equation (5) subject to the boundary conditions (6).

Solution: To do so, we substitute $u(x,t) = X(x)T(t)$ into the equation to obtain

$$\frac{\partial^2}{\partial t^2}(X(x)T(t)) = a^2 \frac{\partial^2}{\partial x^2}(X(x)T(t)) \qquad \Longrightarrow \qquad X(x)T''(t) = a^2 T(t)X''(x)$$

To separate the variables, we then divide throughout by $a^2 X(x)T(t)$:

$$\frac{X(x)T''(t)}{a^2 X(x)T(t)} = \frac{a^2 T(t)X''(x)}{a^2 X(x)T(t)}$$

This in turn simplifies to

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}$$

As a result, there must be a constant λ such that

$$\frac{T''}{a^2 T} = \lambda \qquad \text{and} \qquad \frac{X''}{X} = \lambda$$

These in turn reduce to the differential equations

$$T'' = \lambda a^2 T \qquad \text{and} \qquad X'' = \lambda X$$

If $\lambda > 0$, however, the oscillations would become arbitrarily large in amplitude, which is not physically possible. Thus, we assume that λ is negative, which is to say that $\lambda = -\omega^2$ for some number ω. As a result, we have

$$T'' = -a^2\omega^2 T \qquad \text{and} \qquad X'' = -\omega^2 X$$

Both equations are harmonic oscillators, so that the general solutions are

$$T(t) = A_1 \cos(a\omega t) + B_1 \sin(a\omega t) \qquad \textit{and} \qquad X(x) = A_2 \cos(\omega x) + B_2 \sin(\omega x)$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

Let's now concentrate on $X(x)$. The boundary conditions (6) imply that

$$u(0,t) = X(0)T(t) = 0 \qquad \textit{and} \qquad u(l,t) = X(l)T(t) = 0$$

If we let $T(t) = 0$, then we will obtain the solution $u(x,t) = 0$ for all t. This is called the *trivial solution* since it is the solution corresponding to the string not moving at all. To *avoid the trivial solution*, we thus assume that

$$X(0) = 0 \qquad \textit{and} \qquad X(l) = 0$$

However, $X(x) = A_2 \cos(\omega x) + B_2 \sin(\omega x)$, so that $X(0) = 0$ implies that

$$0 = A_2 \cos(0) + B_2 \sin(0) = A_2$$

Thus, $A_2 = 0$ and $X(x) = B_2 \sin(\omega x)$. The boundary condition $X(l) = 0$ then implies that

$$B_2 \sin(\omega l) = 0$$

If we let $B_2 = 0$, then we again obtain the trivial solution. To avoid the trivial solution, we let $\sin(\omega l) = 0$, which in turn implies that

$$\omega l = n\pi$$

for any integer n. Thus, there is a solution for $\omega_n = n\pi/l$ for each value of n, which means that

$$X_n(x) = B_2 \sin\left(\frac{n\pi}{l}x\right)$$

is a solution to the vibrating string equation for each n. Consequently, for each integer n there is a separated solution of the form

$$u_n(x,t) = \left[A_1 \cos\left(\frac{an\pi}{l}t\right) + B_1 \sin\left(\frac{an\pi}{l}t\right)\right] B_2 \sin\left(\frac{n\pi}{l}x\right) \qquad (7)$$

Check your Reading: Where did the $an\pi/l$ come from in the final form of the separated solution?

Linearity and Fourier Series

We say that a partial differential equation is *linear* if the linear combination of any two solutions is also a solution. For example, suppose that $p(x,t)$ and $q(x,t)$ are both solutions to the *heat equation*—i.e., suppose that

$$\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} \qquad \textit{and} \qquad \frac{\partial q}{\partial t} = k\frac{\partial^2 q}{\partial x^2} \qquad (8)$$

A linear combination of p and q is of the form $u(x,t) = Ap(x,t) + Bq(x,t)$ where A, B are both constants. Moreover,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(Ap(x,t) + Bq(x,t)\right) = A\frac{\partial p}{\partial t} + B\frac{\partial q}{\partial t}$$

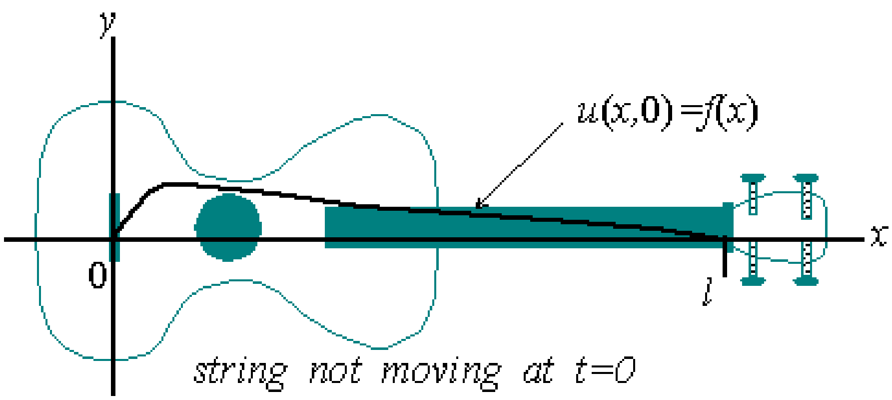
so that (8) implies that

$$\frac{\partial u}{\partial t} = A\frac{\partial p}{\partial t} + B\frac{\partial q}{\partial t} = Ak\frac{\partial^2 p}{\partial x^2} + Bk\frac{\partial^2 q}{\partial x^2} = k\frac{\partial^2}{\partial x^2}\left(Ap(x,t) + Bq(x,t)\right)$$

That is, the linear combination $u(x,t) = Ap(x,t) + Bq(x,t)$ is also a solution to the heat equation, and consequently, we say that the heat equation is a *linear* partial differential equation.

Suppose now that a linear partial differential equation has both boundary conditions and initial conditions, where *initial conditions* are constraints on the solution and its derivatives at a fixed point in time. Then a complete solution to the partial differential equation can often be obtained from the *Fourier series* decompositions of the initial conditions.

For example, let us suppose that the vibrating string in example 5 is *plucked* at time $t = 0$, which is to say that it is released from rest at time $t = 0$ with an initial shape given by the graph of the function $y = f(x)$:



Then the *initial conditions* for the vibrating string are

$$u(x,0) = f(x) \qquad \textit{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

Let's apply the initial conditions to the separated solution (7). The initial condition $u_t(x,0) = 0$ implies that $X(x)T'(0) = 0$, so that to avoid the trivial solution we suppose that $T'(0) = 0$. Thus,

$$0 = T'(0) = -a\omega A_1 \sin(0) + B_1 a\omega \cos(0) = B_1$$

As a result, we must have $T(t) = A_1 \cos(an\pi t/l)$, and if we define $b_n = A_1 B_2$, then (7) reduces to

$$u_n(x,t) = b_n \cos\left(\frac{an\pi}{l}t\right) \sin\left(\frac{n\pi}{l}x\right) \tag{9}$$

As will be shown in the exercises, the 1 dimensional wave equation is linear. Thus, if $u_j(x,t)$ and $u_k(x,t)$ are solutions (9) for integers j and k, then $u_j(x,t) + u_k(x,t)$ is also a solution. In fact, the sum all possible solutions, which is the sum of all solutions for any positive integer value of n, is a solution called the *general solution.* That is, the general solution to the 1 dimensional wave equation with the given boundary and initial conditions is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{an\pi}{l}t\right) \sin\left(\frac{n\pi}{l}x\right) \tag{10}$$

Hence, the only task left is that of determining the values of the constants b_n. However, (10) implies that

$$u(x,0) = \sum_{n=1}^{\infty} b_n \cos(0) \sin\left(\frac{n\pi}{l}x\right)$$

and since $u(x,0) = f(x)$, this reduces to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

As a result, if $f(x)$ is continuous and if $f(0) = f(l)$, then the constants b_n are the *Fourier Sine coefficients* of $f(x)$ on $[0,l]$, which are given by

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx \tag{11}$$

For more on Fourier series and their relationship to partial differential equations, see the Maple worksheet associated with this section.

EXAMPLE 6 What is the solution to the vibrating string problem for a 2 foot long string which is initially at rest and which has an initial shape that is the same as the graph of the function

$$u(x,0) = \frac{1}{12} - \frac{|x-1|}{12}$$

Solution: We begin by finding the Fourier coefficients b_n, which according to (11) are for an $l = 2$ foot long string given by

$$b_n = \frac{2}{2}\int_0^2 \left(\frac{1}{12} - \frac{|x-1|}{12}\right) \sin\left(\frac{n\pi}{2}x\right) dx$$

:Evaluating using the computer algebra system *Maple* then yields

$$b_n = \frac{2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)}{3n^2\pi^2}$$

However, since n is an integer, $\sin(n\pi) = 0$ for all n. Thus, b_n reduces to

$$b_n = \frac{2\sin\left(\frac{n\pi}{2}\right)}{3n^2\pi^2}$$

But $\sin\left(\frac{n\pi}{2}\right) = 0$ when n is even, so that $b_0 = b_2 = \ldots = b_{2n} = 0$. Thus, we only have odd coefficients of the form

$$b_1 = \frac{2\sin\left(\frac{\pi}{2}\right)}{3\cdot 1^2\cdot \pi^2}, \qquad b_3 = \frac{2\sin\left(\frac{3\pi}{2}\right)}{3\cdot 3^2\cdot \pi^2}, \quad b_5 = \frac{2\sin\left(\frac{5\pi}{2}\right)}{3\cdot 5^2\cdot \pi^2}, \ldots$$

which simplify to

$$b_1 = \frac{2(1)}{3\cdot 1^2\cdot \pi^2}, \qquad b_3 = \frac{2(-1)}{3\cdot 3^2\cdot \pi^2}, \quad b_5 = \frac{2(1)}{3\cdot 5^2\cdot \pi^2}, \ldots$$

Odd numbers are of the form $2n+1$ for $n = 0, 1, \ldots$. Thus, we have

$$b_{2n+1} = \frac{2(-1)^n}{3\pi^2(2n+1)^2}$$

and the solution (10) is of the form

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2(-1)^n}{3\pi^2(2n+1)^2} \cos\left(\frac{a(2n+1)\pi}{l}t\right) \sin\left(\frac{(2n+1)\pi}{l}x\right)$$

The Fourier series (10) is known as the *Harmonic Series* in music theory. Indeed, if we write the Fourier series in expanded form

$$u(x,t) = b_1\cos\left(\frac{a\pi}{l}t\right)\sin\left(\frac{\pi}{l}x\right) + b_2\cos\left(\frac{2a\pi}{l}t\right)\sin\left(\frac{2\pi}{l}x\right) + b_3\cos\left(\frac{3a\pi}{l}t\right)\sin\left(\frac{3\pi}{l}x\right) + \ldots$$

then the first term is known as the *fundamental,* which corresponds to the string shape of $y = \sin(\pi x/l)$, which is fixed at $x = 0$ and $x = l$, oscillating at an amplitude of b_1. The oscillations themselves have a frequency of

$$f_1 = \frac{a\pi}{l}\ \frac{rad}{\sec} \cdot \frac{1 cycle}{2\pi\ rad} = \frac{a}{2l}\ \frac{cycles}{\sec}$$

A "cycle per second" is known as a *Hertz* and recall that $a = k/\mu$, so that

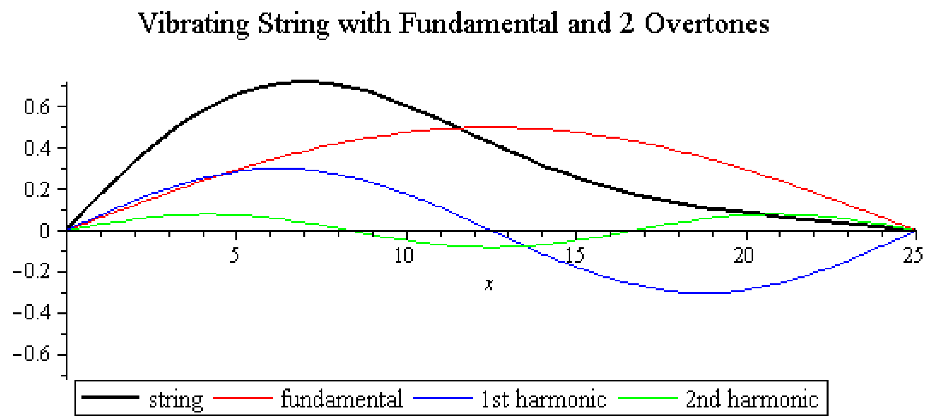
$$f_1 = \frac{k}{2\mu l}\ Hz$$

Thus, increases in tension k cause the fundamental pitch to rise, while lengthening the string lowers the pitch. A heavier string (larger μ) has a lower pitch than a lighter string.

The second term in the Harmonic Series of the string oscillates at an amplitude b_2 with twice the frequency of the fundamental,

$$f_2 = \frac{2a\pi}{l} = 2f_1.$$

It is known as the *first harmonic* or *first overtone* of the string, and it corresponds to the oscillation of a string shape $y = \sin(2\pi x/l)$ that is fixed at $x = 0$, $x = l/2$, and $x = l$ – i.e., half as long as the fundamental. Similarly, the third term is the *second harmonic*, which oscillates at a frequency of $f_3 = 3f_1$ and which corresponds to oscillations at amplitude b_3 of sinusoidal shapes a third as long as the fundamental.



For example, if the string is at a length, tension, and mass so as to oscillate with a frequency of 440 hz ("A" above middle "C"), then we also hear a pitch of $f_2 = 880$ hz (an octave above the fundamental), a pitch of $f_3 = 3(440)$ hz (an octave and a fifth above the fundamental) and so on.

Exercises

Show that the given function is a solution to the given partial differential equation. Assume that k, ω, a, and c are constants.

1. $u(x,y) = x^3 - 3xy^2$ is a solution to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
2. $u(x,y) = 3x^2 y - y^3$ is a solution to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
3. $u(x,t) = 2t + x^2$ is a solution to $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$
4. $u(x,t) = x^2 + t^2$ is a solution to $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$
5. $u(x,y) = e^x \sin(y)$ is a solution to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
6. $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ is a solution to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
7. $u(x,t) = e^{-\omega^2 kt}\cos(\omega x)$ is a solution to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$
8. $u(x,t) = \sin(\omega x)\sin(a\omega t)$ is a solution to $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$
9. $u(x,t) = f(x+ct)$ is a solution to $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$
10. $u(x,t) = f(x-ct)$ is a solution to $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

Find the separated solution to each of the following partial differential equations.

Assume that k, a, c, and τ are constant.

11.	$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}$	12.	$\frac{\partial u}{\partial t}=-k\frac{\partial u}{\partial x}$
13.	$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$	14.	$\frac{\partial u}{\partial x}=-2x\frac{\partial u}{\partial y}$
15.	$F_x+e^{-x}F_y=0$	16.	$F_x+3x^2F_y=0$
17.	$u_x+u_t=u$	18.	$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=u$
19.	$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$	20.	$u_t=u_{xx}+u$
21.	$\frac{\partial^2 V}{\partial x^2}-\tau\frac{\partial V}{\partial t}-V=0$	22.	$u_t=u_{xx}-u$
23.	$\frac{\partial^2 u}{\partial t^2}-\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}=0$	24.	$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}=0$

25. Show that

$$u(x,t)=\frac{1}{\sqrt{t}}e^{-x^2/(4t)}$$

is a solution to the heat equation $u_t=u_{xx}$.

26. Show that $u(x,y,z)=\left(x^2+y^2+z^2\right)^{1/2}$ is a solution to the *3 dimensional Laplace equation*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

27. Let $i^2=-1$ and suppose that $u(x,y)$ and $v(x,y)$ are such that

$$(x+iy)^2=u(x,y)+i\,v(x,y)$$

Find u and v and show that both satisfy *Laplace's equation*—that is, that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \textit{and} \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

In addition, show that u and v satisfy the *Cauchy-Riemann Equations*

$$u_x=v_y, \qquad u_y=-v_x$$

28. Let $i^2=-1$ and suppose that $u(x,y)$ and $v(x,y)$ are such that

$$(x+iy)^4=u(x,y)+i\,v(x,y)$$

Find u and v and show that both satisfy *Laplace's equation*—that is, that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \textit{and} \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

In addition, show that u and v satisfy the *Cauchy-Riemann Equations*

$$u_x=v_y, \qquad u_y=-v_x$$

29. Suppose that a large population of micro-organisms (e.g., bacteria or plankton) is distributed along the x-axis. If $u(x,t)$ is the population per unit length at location x and at time t, then $u(x,t)$ satisfies a *diffusion equation* of the form

$$\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial x^2}+ru$$

where μ is the rate of dispersal and r is the birthrate of the micro-organisms. If μ and r are positive constants, then what is a separated solution of this diffusion equation? (adapted from *Mathematical Models in Biology,* Leah Edelstein-Keshet, Random House, 1988, p. 441).

30. Suppose that t denotes time and x denotes the age of a cell ina given population of cells, and let

$$u(x,t)\,dx = \begin{array}{c} number\ of\ cells\ whose \\ age\ at\ time\ t\ is \\ between\ x\ and\ x+dx \end{array}$$

Then $u(x,t)$ is the *cell density per unit age* at time t, and given appropriate assumptions, it satisfies

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x} = d_0 \frac{\partial^2 u}{\partial x^2}$$

where v_0 and d_0 are positive constants. What is the separated solution to this equation? (adapted from *Mathematical Models in Biology,* Leah Edelstein-Keshet, Random House, 1988, p. 466).

31. Find the separated solution of the *telegraph equation with zero self inductance:*

$$\frac{\partial^2 u}{\partial x^2} = RC\frac{\partial u}{\partial t} + RSu$$

Here $u(x,t)$ is the electrostatic potential at time t at a point x units from one end of a transmission line, and R, C, and S are the resistance, capacitance, and leakage conductance per unit length, respectively.

32. If $V(x,t)$ is the membrane voltage at time t in seconds and at a distance x from the distal (i.e., initial) end of a uniform, cylindrical, unbranched section of a dendrite, then $V(x,t)$ satisfies

$$\frac{d}{4R_i}\frac{\partial^2 V}{\partial x^2} = C_m \frac{\partial V}{\partial t} + \frac{1}{R_m} V \tag{12}$$

where d is the diameter of the cylindrical dendritic section, R_i is the resistivity of the intracellular fluid, C_m is the membrane capacitance, and R_m is the membrane resistivity. Find a separated solution to (12) given that C_m, R_m, and R_i are positive constants.

33. In Quantum mechanics, a particle moving in a straight line is said to be in a *state* $\psi(x,t)$ if

$$\int_a^b |\psi(x,t)|^2\,dx$$

represents the probability of the particle being in the interval $[a,b]$ on the line at time t. If a subatomic particle is traveling in a straight line close to the speed of light, then it's state satisfies the *one dimensional Klein-Gordon Equation*

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = \lambda\psi$$

where $\lambda > 0$ is constant. Find the separated solution of the one dimensional Klein-Gordon equation.

34. If a subatomic particle is traveling in a straight line much slower than the speed of light and no forces are acting on that particle, then its state (as explained in problem 33) satisfies the *one dimensional Schrödinger equation of a single free particle.*

$$\frac{\partial \psi}{\partial t} = -i\frac{\partial^2 \psi}{\partial x^2} \tag{13}$$

where $i^2 = -1$. Find the separated solution of (13) (Hint: you will need to use Euler's identity

$$e^{it} = \cos(t) + i\sin(t)$$

35. Show that if $u(x,t)$ and $v(x,t)$ are both solutions to the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

then so also is the function $w(x,t) = Au(x,t) + Bv(x,t)$ where A and B are constants. What does this say about the wave equation?

36. Show that if $u(x,y)$ and $v(x,y)$ are both solutions to *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

then so also is the function $w(x,y) = Au(x,y) + Bv(x,y)$ where A and B are constants. What does this tell us about Laplace's equation?

37. Suppose that the initial conditions for the guitar string in example 6 are

$$u(x,0) = \sin\left(\frac{x}{2}\right) \quad \textit{and} \quad \frac{\partial u}{\partial t}(x,0) = 0$$

What are the coefficients b_n in the solution (10) for these initial conditions?

38. Solve the vibrating string problem for the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \textit{and} \qquad \frac{\partial u}{\partial x}(l,t) = 0$$

and for the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$.

39. Heat Equation I: Find the general solution to the *heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u(0,t) = 0 \qquad u(\pi,t) = 0$$

40. Heat Equation II: If the initial condition is $u(x,0) = \pi x - x^2$, then what are the Fourier coefficients in the general solution found in exercise 39?

41. Laplace's Equation I: Find the general solution to the *Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions

$$u(0, y) = 0 \qquad u(\pi, y) = 0$$

42. Laplace's Equation II: If the initial conditions are $u(x, 0) = \sin(x/2)$ and $u_y(x, 0) = 0$, then what are the Fourier coefficients in the general solution found in exercise 41?

43. Write to Learn: In a short essay, explain in your own words why an equation of the form

$$f(x, y) = g(t)$$

implies that both $f(x, y)$ and $g(t)$ are constant. (x, y, and t are both independent variables).

44. ***What is a separated solution of the *2*-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2}$$

45. Find a separated solution of the following *nonlinear wave equation:*

$$\frac{\partial u}{\partial t} = cu\frac{\partial u}{\partial x}$$