Partial Derivatives

Partial Derivatives

Just as derivatives can be used to explore the properties of functions of 1 variable, so also derivatives can be used to explore functions of 2 variables. In this section, we begin that exploration by introducing the concept of a *partial derivative* of a function of 2 variables.

In particular, we define the *partial derivative* of $f(x,y)$ with respect to x to be

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

when the limit exists. That is, we compute the derivative of $f(x,y)$ as if x is the variable and all other variables are held constant. To facilitate the computation of partial derivatives, we define the operator

$$\frac{\partial}{\partial x} = \text{“The partial derivative with respect to } x\text{”}$$

Alternatively, other notations for the partial derivative of f with respect to x are

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \partial_x f(x,y)$$

EXAMPLE 1 Evaluate f_x when $f(x,y) = x^2 y + y^2$.

Solution: To do so, we write

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2 y + y^2\right) = \frac{\partial}{\partial x} x^2 y + \frac{\partial}{\partial x} y^2$$

and then we evaluate the derivative as if y is a constant. In particular,

$$f_x(x,y) = y\frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial x} y^2 = y \cdot 2x + 0$$

That is, y factors to the front since it is considered constant with respect to x. Likewise, y^2 is considered constant with respect to x, so that its derivative with respect to x is 0. Thus, $f_x(x,y) = 2xy$.

Likewise, the *partial derivative* of $f(x,y)$ *with respect to* y is defined

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

when the limit exists. That is, we evaluate f_y as if y is varying and all other quantities are constant. Moreover, we also define the operator

$$\frac{\partial}{\partial y} = \text{"The partial derivative with respect to } y\text{"}$$

and we often use other notations for $f_y(x, y)$:

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \partial_y f(x, y)$$

EXAMPLE 2 Find f_x and f_y when $f(x, y) = y \sin(xy)$

Solution: To find f_x, we use the chain rule

$$f_x = y \frac{\partial}{\partial x} \sin(xy) = y \cos(xy) \frac{\partial}{\partial x} xy = y^2 \cos(xy)$$

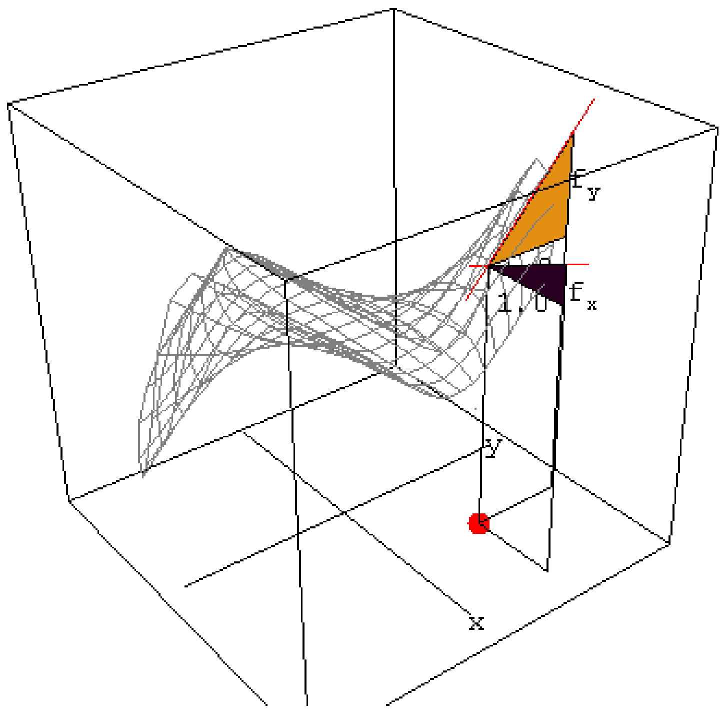
However, to find f_y, we begin with the product rule:

$$f_y = \frac{\partial}{\partial y} [y \sin(xy)] = \left(\frac{\partial}{\partial y} y \right) \sin(xy) + y \frac{\partial}{\partial y} \sin(xy)$$

We then use the chain rule to evaluate $\partial_y \sin(xy)$:

$$\begin{aligned} f_y &= \sin(xy) + y \cos(xy) \frac{\partial}{\partial y}(xy) \\ &= \sin(xy) + xy \cos(xy) \end{aligned}$$

If $y = q$ for some constant q, then $f(x, q)$ is a function of x, and $f_x(x, q)$ is the slope of the tangent line to the curve $z = f(x, q)$ in the $y = q$ plane. Similarly, $f_y(p, y)$ for p constant is the slope of a tangent line to the curve $z = f(p, y)$ in the $x = p$ plane.



Drag to rotate. Drag red point to change the point of tangency.
Shift-drag to resize or to tilt z-axis.

That is, $f_x(x, y)$ is the slope of a tangent line to $z = f(x, y)$ parallel to the xz-plane, while $f_y(x, y)$ is the slope of a tangent line to $z = f(x, y)$ in the yz-plane, an idea we will explore more fully in a later section.

blue*EXAMPLE 3* blackFind f_x and f_y when

$$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

Solution: To find f_x, we begin with the chain rule

$$f_x = \frac{\partial}{\partial x}\tan^{-1}(\text{input}) = \frac{1}{(\text{input})^2 + 1}\frac{\partial}{\partial x}(\text{input})$$

where the input is y/x. Writing the input as $x^{-1}y$ and substituting then yields

$$f_x = \frac{1}{(x^{-1}y)^2 + 1}\frac{\partial}{\partial x}\left(x^{-1}y\right) = \frac{-x^{-2}y}{x^{-2}y^{-2} + 1}$$

To simplify this expression, we multiply the numerator and denominator by x^2:

$$f_x = \frac{x^2\left(-x^{-2}y\right)}{x^2\left(x^{-2}y^{-2}+1\right)} = \frac{-x^2x^{-2}y}{x^2x^{-2}y^2+x^2} = \frac{-y}{y^2+x^2}$$

To find f_y, we again begin with the chain rule,

$$f_y = \frac{\partial}{\partial y}\tan^{-1}(\text{input}) = \frac{1}{(\text{input})^2+1}\frac{\partial}{\partial y}(\text{input})$$

where the input is y/x. The result is that

$$f_y = \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{\left(\frac{y}{x}\right)^2+1}\left[\frac{1}{x}\frac{\partial}{\partial y}(y)\right]$$

which simplifies to

$$f_y = \frac{1}{\left(\frac{y^2}{x^2}+1\right)x} = \frac{x}{\left(\frac{y^2}{x^2}+1\right)x^2} = \frac{x}{x^2+y^2}$$

Check your Reading: What happend to the expression x^2x^{-2} in example 3?

Interpretations of the Partial Derivative

Analogous to a function of 2 variables, we define a *function of 3 variables* is a mapping that assigns one and only one real number to each point in a subset of 3 dimensional space. It follows that a function of three variables is of the form

$$F(x,y,z) = \text{“expression in } x, y, \text{ and } z\text{”}$$

Partial derivatives of functions of 3 variables are defined analogously to partial derivatives of functions of two variables (see the exercises). Thus, to differentiate a function of 3 variables $F(x,y,z)$ with respect to x, we differentiate as if y and z are constants. The partial derivatives F_y and F_z are defined similarly , and correspondingly, to calculate F_y and F_z, we differentiate with respect to y and z, respectively.

EXAMPLE 4 Find the first partial derivatives of $F(x,y,z) = x^3 + 3xyz + z^2$.

Solution: To compute f_x, we treat y and z as if they were constant:

$$\begin{aligned} F_x &= \frac{\partial}{\partial x}\left(x^3+3xyz+z^2\right) \\ &= \frac{\partial}{\partial x}x^3+3yz\frac{\partial}{\partial x}x+\frac{\partial}{\partial x}z^2 \\ &= 3x^2+3yz \end{aligned}$$

Likewise, F_y follows from treating x and z like constants,

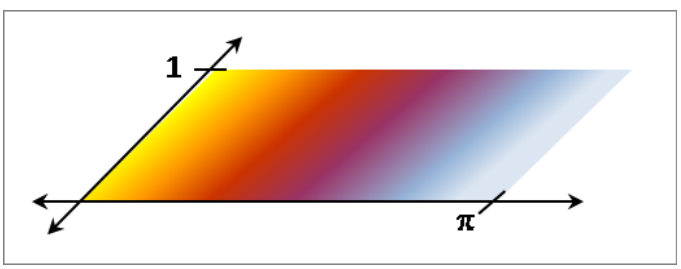
$$F_y=\frac{\partial}{\partial y}\left(x^3+3xyz+z^2\right)=0+3xz+0=3xz$$

and F_z follows from treating x and y like constants.

$$F_z=\frac{\partial}{\partial z}\left(x^3+3xyz+z^2\right)=3xy+2z$$

In many applications, functions of three variables occur in the form $u(x,y,t)$, where t is a measure of *time.* In such examples, the definition of the partial derivative u_t implies that it is *the rate of change* of $u(x,y,t)$ with respect to t. Indeed, partial derivatives often occur in applications as a rate of change of a given output with respect to only one of several inputs.

EXAMPLE 5 A rectangular sheet of metal with a length of π feet and a width of 1 foot has its left section placed in an oven and its rightmost extent placed in liquid nitrogen.



Upon being removed from the oven and nitrogen, its temperature u in $^\circ F$ at time t in seconds and at a point (x,y) on the sheet is given by

$$u(x,y,t)=75+300e^{-0.2t}\cos(x)\cosh(y)$$

How fast is the temperature of the sheet changing with respect to time at the point $(0,0)$? At the point $(\pi,0)$? How do the rates compare?

Solution: The rate of change of u with respect to t is given by

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\left(75+300e^{-0.2t}\cos(x)\cosh(y)\right)=-60e^{-0.2t}\cos(x)\cosh(y)$$

Thus, at $(0,0)$, the time rate of change of temperature is

$$\frac{\partial u}{\partial t}(0,0,t)=-60e^{-0.2t}\cos(0)\cosh(0)=-60e^{-0.2t}\ {}^{\circ}F\ per\ \sec$$

while at $(\pi,0)$, the time rate of change of temperature is

$$\frac{\partial u}{\partial t}(\pi,0,t)=-60e^{-0.2t}\cos(\pi)\cosh(0)=60e^{-0.2t}\ {}^{\circ}F\ per\ \sec$$

This means that the temperature increases to $75^{\circ}F$ over time at the point $(\pi,0)$, while it decreases down to $75^{\circ}F$ at $(0,0)$. **Note**: this fits well with the fact that the temperature initially at $(0,0)$ is $375^{\circ}F$, while the tmperature initially at $(\pi,0)$ is $-225^{\circ}F$.

Check your Reading: After a long period of time, what will be the approximate temperature of the metal sheet at every point on the sheet?

Second Derivatives

The second partial derivative of f with respect to x is denoted f_{xx} and is defined

$$f_{xx}(x,y)=\frac{\partial}{\partial x}f_x(x,y)$$

That is, f_{xx} is the derivative of the first partial derivative f_x. Likewise, the second partial derivative of f with respect to y is denoted f_{yy} and is defined

$$f_{yy}(x,y)=\frac{\partial}{\partial y}f_y(x,y)$$

Finally, the *mixed* partial derivatives are denoted f_{xy} and f_{yx}, respectively, and are defined

$$f_{xy}(x,y)=\frac{\partial}{\partial y}f_x(x,y)\qquad\text{and}\qquad f_{yx}=\frac{\partial}{\partial x}f_y(x,y)$$

Collectively, f_{xx}, f_{yy}, f_{xy}, and f_{yx} are known as the *second partial derivatives* of $f(x,y)$. Moreover, we sometimes denote the second partial derivatives in the form

$$f_{xx}=\frac{\partial^2 f}{\partial x^2},\qquad f_{xy}=\frac{\partial^2 f}{\partial x\partial y},\qquad f_{yy}=\frac{\partial^2 f}{\partial y^2},\qquad f_{yx}=\frac{\partial^2 f}{\partial y\partial x}$$

EXAMPLE 6 Find the second partial derivatives of

$$f(x,y) = x^3 + 3x^2y^2$$

Solution: The first partial derivatives are $f_x = 3x^2 + 6xy^2$ and $f_y = 6x^2y$. As a result, we have

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x} f_x(x,y) = \frac{\partial}{\partial x}\left(3x^2 + 6xy^2\right) = 6x + 6y^2 \\
f_{yy}(x,y) &= \frac{\partial}{\partial y} f_y(x,y) = \frac{\partial}{\partial y}\left(6x^2y\right) = 6x^2 \\
f_{xy}(x,y) &= \frac{\partial}{\partial y} f_x(x,y) = \frac{\partial}{\partial y}\left(3x^2 + 6xy^2\right) = 12xy \\
f_{yx}(x,y) &= \frac{\partial}{\partial x} f_y(x,y) = \frac{\partial}{\partial x}\left(6x^2y\right) = 12xy
\end{aligned}$$

Notice that the mixed partial derivatives are the same. Indeed, the mixed partials are always the same for "nice" functions, as is stated below in *Clairaut's theorem.*

Clairaut's Theorem: If f is defined on a neighborhood of (p,q) and if f_{xy} and f_{yx} are continuous on that neighborhood, then

$$f_{xy}(p,q) = f_{yx}(p,q)$$

There are functions that do not satisfy the hypotheses of theorem 3.1 for which the mixed partials are not the same at some point (see exercise 46). However, our focus will be on functions with continuous second partial derivatives, in which case the mixed partials are the same (a proof of Clairaut's theorem is given in chapter 4).

EXAMPLE 7 Find f_{yx} and f_{xy} for $f(x,y) = x\sin(xy)$

Solution: The first partial derivatives are

$$f_x = \sin(xy) + xy\cos(xy) \qquad \text{and} \qquad f_y = x^2\cos(xy)$$

The product rule thus implies that

$$f_{yx} = \frac{\partial}{\partial x}\left(x^2\cos(xy)\right) = 2x\cos(xy) - x^2y\sin(xy)$$

Now let's compute f_{xy} (and thus confirm theorem 3.1):

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left(\sin\left(xy\right) + xy\cos\left(xy\right)\right) \\ &= x\cos\left(xy\right) + x\cos\left(xy\right) + xy\left(-\sin\left(xy\right)\frac{\partial}{\partial y}xy\right) \\ &= 2x\cos\left(xy\right) - x^2 y\sin\left(xy\right) \end{aligned}$$

Check your Reading: If $f(x,y,z)$ is infinitely differentiable in each variable, then is $f_{xz} = f_{zx}$?

Higher Derivatives

Higher partial derivatives are defined similarly. For example, the third derivative of f with respect to x is the partial derivative with respect to x of the second derivative f_{xx}. That is,

$$f_{xxx}(x,y) = \frac{\partial}{\partial x} f_{xx}(x,y)$$

Similarly, f_{xxy} is defined

$$f_{xxy}(x,y) = \frac{\partial}{\partial y} f_{xx}(x,y)$$

In operator notation, the partial derivative of f for m times with respect to x and n times with respect to y is denoted by

$$\frac{\partial^{m+n} f}{\partial x^m\, \partial y^n}$$

The $m+n$ partial derivatives of $f(x,y)$ are then defined in terms of the previous partial derivatives as

$$\frac{\partial^{m+n} f}{\partial x^m\, \partial y^n} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^{m+n-2} f}{\partial x^{m-1}\partial y^{n-1}}\right)$$

when $f(x,y)$ and its partial derivatives are continuous on a region through the $m+n$ order.

EXAMPLE 8 Find f_{xxyy} if $f(x,y) = x^4 y^4$.

Solution: It is easy to show that $f_{xx} = 12x^2y^4$. Thus,

$$f_{xxy} = \frac{\partial}{\partial y} f_{xx} = \frac{\partial}{\partial y} 12x^2y^4 = 48x^2y^3$$

and similarly,

$$f_{xxyy} = \frac{\partial}{\partial y} f_{xxy} = \frac{\partial}{\partial y} 48x^2y^3 = 144x^2y^2$$

Moreover, we usually assume that f is sufficiently smooth at all points where partial derivatives are defined so that mixed partials are independent of the order of differentiation. Indeed, notice that if $f(x,y) = x^4y^4$, then

$$f_{xyx} = \frac{\partial}{\partial x} f_{xy} = \frac{\partial}{\partial x} 16x^3y^3 = 48x^2y^3$$

which is the same as f_{xxy} in example 8. In addition, $f_{xyxy} = 144x^2y^2 = f_{xxyy}$.

Exercises

Find $f_x(x,y)$ and $f_y(x,y)$ for each of the following:

1.	$f(x,y) = x^2 + y^3$	2.	$f(x,y) = x^2 + 2xy + y^3$
3.	$f(x,y) = (x+2y)^2$	4.	$f(x,y) = \left(x^2 + 2y\right)^2$
5.	$f(x,y) = x\sin(y)$	6.	$f(x,y) = e^x \ln\left(y^2+1\right)$
7.	$f(x,y) = \exp\left(-x^2 - y^2\right)$	8.	$f(x,y) = \tan^{-1}(xy)$
9.	$f(x,y) = x\cos(xy)$	10.	$f(x,y) = x\sin(xy)$
11.	$f(x,y) = y^x$	12.	$f(x,y) = x^y$
13.	$f(x,y) = \frac{x}{x^2+y^2}$	14.	$f(x,y) = \int_x^y \sin\left(t^2\right) dt$

Find f_{xx}, f_{xy}, f_{yx}, and f_{yy} for each of the following. Then show that the mixed partials are the same.

15.	$f(x,y) = x^2 + y^3$	16.	$f(x,y) = x^2 + 2xy + y^3$
17.	$f(x,y) = (x+2y)^2$	18.	$f(x,y) = \left(x^2 + 2y\right)^2$
19.	$f(x,y) = x\sin(y)$	20.	$f(x,y) = e^x \ln\left(y^2+1\right)$
21.	$f(x,y) = x\cos(xy)$	22.	$f(x,y) = \tan^{-1}(xy)$
23.	$f(x,y) = y^x$	24.	$f(x,y) = x^y$

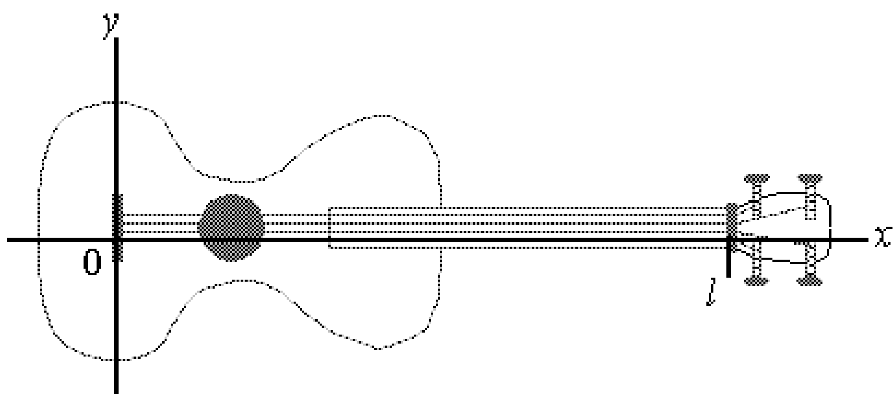
Find the indicated derivative of the given function:

25. f_{xxy} for $f(x,y) = x^2 + y^3$
26. f_{xxy} for $f(x,y) = x^2 + 2xy + y^3$
27. f_{xyx} for $f(x,y) = (x+2y)^2$
28. f_{yxy} for $f(x,y) = \left(x^2 + 2y\right)^2$
29. f_{xxyy} for $f(x,y) = x\sin(y)$
30. $f_{xxxxxxxy}$ for $f(x,y) = e^x \ln\left(y^2+1\right)$
31. f_{xxxy} for $f(x,y) = y\cos(xy)$
32. f_{xyy} for $f(x,y) = \sin(x)\tan(xy)$

33. A vibrating string has a displacement $y = u(x,t)$ in cm at a distance x in cm from one end and at time t in seconds, where

$$u(x,t) = 2\sin(120\pi(x-t))$$

How fast (in units of cm per sec) is the string vibrating at a horizontal distance $x = 1.2$ cm from one end at time $t = 2$ seconds? At time $t = 3$ seconds?
34. Suppose that a string is attached at its endpoints $x = 0$ and $x = l$, for some number l.



Suppose also that $y = u(x,t)$ models the displacement at x in $[0,l]$ of the string at time t, where

$$u(x,t) = A\cos(at)\sin\left(\frac{\pi}{l}x\right)$$

with A and a constant.

1. (a) Show that $u(0,t) = u(l,t) = 0$ for all t. How does this relate to $u(x,t)$ being a model of a string?
   (b) What is the rate of change of $u(x,t)$ at $x = l/2$ at any given time?

35. The function $u(x,t) = e^{-t}\sin^2(\pi x) + 32$ models the temperature in $^\circ F$ of a 1 foot long thin rod in which both ends are held at the freezing point at all times t. How fast is the temperature decreasing at the midpoint of the rod when $t = 0$? When $t = 1$? When $t = 2$?
36. The function $u(x,y,t) = 2\sin(3x)\sin(4y)\cos(5t)$ models the displacement u in cm of a vibrating rectangular membrane at time t in seconds and at

a point (x, y) on the membrane. How fast is the displacement of the membrane above the point $(1, 1)$ changing with respect to time at $t = 1$ seconds?

37. It can be shown that an ideal gas with fixed mass has an absolute temperature R, a pressure P, and a volume V that satisfies

$$T = kPV$$

where k is a constant. How fast does the temperature T change with respect to the volume V?

38. The total resistance R produced by two resistors with resistances R_1 and R_2, respectively, satisfies

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

What is the rate of change of the total resistance R with respect to the resistance R_1?

39. If two planets with masses M and m are located at the points (x, y, z) and $(0, 0, 0)$, respectively, then the potential energy of their mutual gravitational attraction is given by

$$\phi(x, y, z) = G \frac{Mm}{\sqrt{x^2 + y^2 + z^2}}$$

where G is the universal gravitational constant. At what rate is the potential energy changing with respect to x? With respect to y?

40. A *Cobb-Douglas* production function is a function of the form $P = bL^{\alpha}K^{\beta}$ where b, α, and β are constants. What is the rate of change of P with respect to L? With respect to P?

41. Suppose we consider $f(x, y) = x^2 + y^2$.

1. (a) What is the slope of $z = f(x, y)$ for $(p, q) = (1, 2)$ in the x-direction? in the y-direction?
   (b) What curve is formed by the intersection of the plane $y = 2$ with the surface $z = x^2 + y^2$? How does it relate to $f_x(1, 2)$?
   (c) What curve is formed by the intersection of the plane $x = 1$ with the surface $z = x^2 + y^2$? How does it relate to $f_y(1, 2)$?

42. Suppose we consider $f(x, y) = x^2 + xy$.

1. (a) What is the slope of $z = f(x, y)$ for $(p, q) = (1, 2)$ in the x-direction? in the y-direction?
   (b) What curve is formed by the intersection of the plane $y = 2$ with the surface $z = x^2 + y^2$? How does it relate to $f_x(1, 2)$?
   (c) What curve is formed by the intersection of the plane $x = 1$ with the surface $z = x^2 + y^2$? How does it relate to $f_y(1, 2)$?

43. If $f(x,y) = g(x) + h(y)$, then what is $\dot{f}_x$ and f_y? What is the equation in x and z of the curve formed by the intersection of $z = f(x,y)$ with the vertical plane $y = q$? with the vertical plane $x = p$? How are these curves related to $f_x(p,q)$ and $f_y(p,q)$, respectively?
44. If $f(x,y) = g(x) + h(y)$, then what is $\dot{f}_x$ and f_y? What is the equation in x and z of the curve formed by the intersection of $z = f(x,y)$ with the vertical plane $y = q$? with the vertical plane $x = p$? How are these curves related to $f_x(p,q)$ and $f_y(p,q)$, respectively?
45. If f_x and f_y both exist, how can the limit

$$\lim_{h\to 0} \frac{f(x+h,y) - f(x,y+h)}{h}$$

be expressed in terms of the 1st partial derivatives of f?
46. Write to Learn: Write a short essay in which you use the following steps to show that

$$f(x,y) = \begin{cases} \frac{x^2y - y^2x}{x+y} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$, that $f_x(x,y)$ and $f_y(x,y)$ are continuous at $(0,0)$, but that

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

1. (a) Show that if $(x,y) \neq (0,0)$, then

$$f(x,y) = \frac{x-y}{\frac{1}{x} + \frac{1}{y}}$$

and correspondingly

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x-y}{\frac{1}{x} + \frac{1}{y}} = 0$$

(b) Show that if $(x,y) \neq (0,0)$, then

$$f_x(x,y) = \frac{y\left(x^2 + 2xy - y^2\right)}{(x+y)^2}$$

and explain why that as a result, we have

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0$$

(c) Define $f_x(0,0) = 0$ and then evaluate

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h}$$

(d) Repeat (b) and (c) beginning with the fact that

$$f_y(x,y) = \frac{x\left(x^2 - 2xy - y^2\right)}{(x+y)^2}$$

The outcome should be that $f_{yx}(0,0)$ is not the same as $f_{xy}(0,0)$.