Practice Test

Chapter 1 Name ______________________

Instructions. Show your work and/or explain your answers..

1. Find the length of each vector and the angle between them.

$$\mathbf{u} = \left\langle \sqrt{2}, \sqrt{6}, 2\sqrt{2} \right\rangle \qquad \text{and} \qquad \mathbf{v} = \langle 0, 0, 1 \rangle$$

Solution: $\mathbf{u} \cdot \mathbf{u} = 2 + 6 + 8 = 16$, $||\mathbf{u}|| = 4$, $||\mathbf{v}|| = 1$, $\mathbf{u} \cdot \mathbf{v} = 2\sqrt{2}$, so

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

Thus, $\theta = \pi/4$.

2. Show that if $\mathbf{u}$ is perpendicular to $\mathbf{v}$, then

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$$

Solution: $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 0$

3. Find a number k for which $\mathbf{u} = \langle 1, 2, 1 \rangle$ is perpendicular to $\mathbf{v} = \langle k, 3, 4 \rangle$.

Solution: $\mathbf{u} \cdot \mathbf{v} = k + 6 + 4 = k + 10$. Thus, $\mathbf{u} \cdot \mathbf{v} = 0$ implies that $k = -10$.

4. Find the area of the triangle whose vertices are $P_1(0,0,0)$, $P_2(1,1,0)$, $P_3(1,1,4)$.

Solution: The vectors are $\mathbf{u} = \langle 1, 1, 0 \rangle$ and $\mathbf{v} = \langle 1, 1, 4 \rangle$. Their cross product is $\mathbf{u} \times \mathbf{v} = \langle 4, -4, 0 \rangle$, so that the area is

$$A = \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{16 + 16 + 0} = 2\sqrt{2}$$

5. Find the equation of the plane through the points $P_1(0,0,0)$, $P_2(2,1,5)$, and $P_3(-1,1,2)$.

Solution: The vectors are $\mathbf{u} = \langle 2, 1, 5 \rangle$ and $\mathbf{v} = \langle -1, 1, 2 \rangle$, so that $\mathbf{u} \times \mathbf{v} = \langle -3, -9, 3 \rangle$. Thus, the equation of the plane is

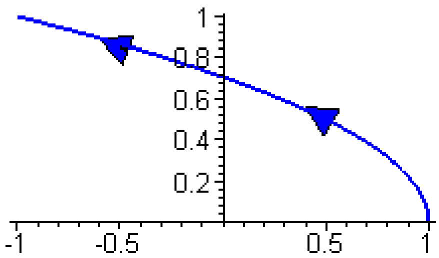
$$-3(x - 0) - 9(y - 0) + 3(z - 0) = 0$$

so that in functional form we have $z = x + 3y$.

6. Find the xy-equation and sketch the graph of the curve

$$\mathbf{r}(t) = \langle \cos(2t), \sin(t) \rangle, \quad t \text{ in } \left[0, \frac{\pi}{2}\right]$$

Solution: Since $\cos(2t) = 1 - 2\sin^2(t)$, we have $x = 1 - 2y^2$. Moreover, $\mathbf{r}(0) = \langle \cos(0), \sin(0) \rangle = (1, 0)$ and $\mathbf{r}(\pi/2) = (\cos(\pi), \sin(\pi/2)) = (-1, 1)$. Thus, the parametrization is the section of the curve $x = 2y^2 - 1$ from $(1, 0)$ to $(-1, 1)$.

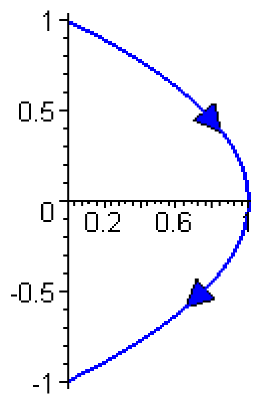


7. Find the cartesian equation of the parametric curve

$$\mathbf{r}(t) = \left\langle \sin^2(t), \cos(t) \right\rangle, \ t \ in \ [0, \pi]$$

Then sketch the curve showing its orientation and its endpoints.

Solution: Since $\cos^2(t) + \sin^2(t) = 1$, we have $y^2 + x = 1$ or $x = 1 - y^2$. Moreover, $\mathbf{r}(0) = \langle 0, 1 \rangle$ and $\mathbf{r}(\pi) = \langle 0, -1 \rangle$.



8. Find the velocity, speed, and acceleration of the curve with parametrization

$$\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle$$

Solution: The velocity is $\mathbf{v}(t) = \langle 1, 2t, 3t^2 \rangle$, the acceleration is $\mathbf{a}(t) = \langle 0, 2, 6t \rangle$, and the speed is

$$v = \sqrt{1 + 4t^2 + 9t^4}$$

9. If a rock is thrown into the air near the earth's surface with initial velocity $\mathbf{v}(0) = \langle 16, 0, 64 \rangle$ feet per second and initial position $\mathbf{r}(0) = \langle 0, 0, 6 \rangle$ feet, then what is the maximum height of the rock if air resistance is ignored?

Solution: Since $\mathbf{a}(t) = \langle 0, 0, -32 \rangle$, we must have $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 0, 0, -32t \rangle + \langle C_1, C_2, C_3 \rangle$. Moreover, $\mathbf{v}(0) = \langle 16, 0, 64 \rangle = \langle 0, 0, 0 \rangle + \langle C_1, C_2, C_3 \rangle$, so that $\mathbf{v}(t) = \langle 16, 0, 64 - 32t \rangle$. Similarly, $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$ and the initial condition lead to

$$\mathbf{r}(t) = \left\langle 16t, 0, 64t - 16t^2 + 6 \right\rangle$$

As a result, $\mathbf{v} \cdot \mathbf{a} = 0$ when $64 - 32t = 0$, or when $t = 2$. Thus, the maximum height is

$$64 \cdot 2 - 16 \cdot 2^2 + 6 = 70\ feet$$

10. The acceleration due to gravity is 12.2 feet per second per second at the surface of Mars. Find the position function $\mathbf{r}(t)$ of an object with initial velocity $\mathbf{v}_0 = \langle 30, 0, 40 \rangle$ and initial position $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$.

Solution: $\mathbf{v}(t) = \int \langle 0, 0, -12.2 \rangle\,dt = \langle 0, 0, -12.2t \rangle + \mathbf{v}_0 = \langle 0, 0, -12.2t \rangle + \langle 30, 0, 40 \rangle = \langle 30, 0, 40 - 12.2t \rangle$. Integrating again yields

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t)\,dt \\ &= \int \langle 30, 0, 40 - 12.2t \rangle\,dt \\ &= \left\langle 30t, 0, 40t - 6.1t^2 \right\rangle + \mathbf{r}_0 \\ &= \left\langle 30t, 0, 40t - 6.1t^2 \right\rangle \end{aligned}$$

11. Find the arclength and the unit tangent vector of the curve

$$\mathbf{r}(t) = \langle 3\sin(t), 5\cos(t), 4\sin(t) \rangle, \ t\ in\ [0, 2\pi]$$

Solution: $\mathbf{v}(t) = \langle 3\cos(t), -5\sin(t), 4\cos(t) \rangle$, so that the speed is

$$\begin{aligned} v &= \sqrt{9\cos^2(t) + 25\sin^2(t) + 16\cos^2(t)} \\ &= \sqrt{25\cos^2(t) + 25\sin^2(t)} \\ &= 5 \end{aligned}$$

Thus, $\mathbf{T}(t) = \langle 3/5\ \cos(t), -\sin(t), 4/5\ \cos(t) \rangle$ and the arclength is

$$L = \int_0^{2\pi} v dt = \int_0^{2\pi} 5 dt = 10\pi$$

12. Find the unit normal $\mathbf{N}$ for the curve

$$\mathbf{r}(t) = \left\langle \sin\left(t^3\right), t^3, \cos\left(t^3\right) \right\rangle$$

Solution: To begin with, $\mathbf{v}(t) = \langle 3t^2 \cos(t^3), 3t^2, 3t^2 \sin(t^3) \rangle$, so that the speed is

$$v = \sqrt{9t^4 \cos^2(t^3) + 9t^4 + 9t^4 \sin^2(t^3)} = 3\sqrt{2}t^2$$

Thus, the unit tangent vector is $\mathbf{T}(t) = \left\langle \cos(t^3)/\sqrt{2}, 1/\sqrt{2}, \sin(t^3)/\sqrt{2} \right\rangle$ and

$$\frac{d\mathbf{T}}{dt} = \left\langle \frac{-3t^2}{\sqrt{2}} \sin\left(t^3\right), 0, \frac{-3t^2}{\sqrt{2}} \cos\left(t^3\right) \right\rangle$$

from which we find that the normal vector is $\mathbf{N}(t) = \langle -\sin(t^3), 0, -\cos(t^3) \rangle$.

13. Find the arclength of the curve

$$\mathbf{r}(t) = \left\langle e^{2t}, t, 2e^t \right\rangle, \quad t \text{ in } [0,1]$$

Solution: $\mathbf{v}(t) = \langle 2e^{2t}, 1, 2e^t \rangle$, so $v = (4e^{4t} + 4e^{2t} + 1)^{1/2} = \left([2e^{2t} + 1]^2 \right)^{1/2} = 2e^{2t} + 1$. Thus,

$$L = \int_0^1 v dt = \int_0^1 \left(2e^{2t} + 1\right) dt = e^2$$

14. Find the length of the astroid

$$\mathbf{r}(t) = \left\langle \cos^3(t), \sin^3(t) \right\rangle, \quad t \text{ in } [0, 2\pi]$$

Solution: The velocity is $\mathbf{v} = \left\langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \right\rangle$, so that the speed is

$$v = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} = \sqrt{9\cos^2(t)\sin^2(t)\left(\cos^2(t) + \sin^2(t)\right)}$$

Thus, the speed is $v = 3|\cos(t)\sin(t)|$, so that the arclength is

$$L = \int_0^{2\pi} v dt = 4\int_0^{\pi/2} 3\sin(t)\cos(t)\, dt = 6$$

15. Find the curvature of the curve

$$\mathbf{r}(t) = \langle \sin(t), \cos(t), \ln|\sec(t)| \rangle$$

Solution: The velocity is $\mathbf{v}(t) = \langle \cos(t), -\sin(t), \tan(t) \rangle$, which implies that the speed is

$$v = \sqrt{\cos^2(t) + \sin^2(t) + \tan^2(t)} = \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)}$$

Thus, $v = \sec(t)$, which implies that the unit tangent vector is

$$\begin{aligned} \mathbf{T}(t) &= \cos(t)\langle\cos(t), -\sin(t), \tan(t)\rangle \\ &= \left\langle\cos^2(t), -\sin(t)\cos(t), \sin(t)\right\rangle \end{aligned}$$

It then follows that the derivative of the unit tangent is

$$\frac{d\mathbf{T}}{dt} = \left\langle 2\cos(t)\sin(t), \sin^2(t) - \cos^2(t), \cos(t)\right\rangle = \langle\sin(2t), -\cos(2t), \cos(t)\rangle$$

As a result, the curvature is

$$\kappa = \frac{1}{\sec(t)}\sqrt{\sin^2(2t) + \cos^2(2t) + \cos^2(t)} = \cos(t)\sqrt{1 + \cos^2(t)}$$