Components of Acceleration

Part 1: Curvature and the Unit Normal

In the last section, we explored those ideas related to velocity–namely, distance, speed, and the unit tangent vector. In this section, we do the same for acceleration by exploring the concepts of linear acceleration, curvature, and the unit normal vector.

Thoughout this section, we will assume that $\mathbf{r}(t)$ parameterizes a smooth curve and is second differentiable in each component. Thus, its unit tangent $\mathbf{T}$ satisfies $\|\mathbf{T}(t)\| = 1$, which implies that $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$. Differentiation yields

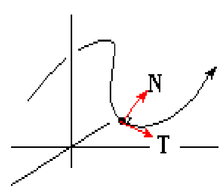
$$\begin{aligned} \frac{d}{dt}(\mathbf{T}(t) \cdot \mathbf{T}(t)) &= \frac{d}{dt}1 \\ \frac{d\mathbf{T}}{dt} \cdot \mathbf{T} + \mathbf{T} \cdot \frac{d\mathbf{T}}{dt} &= 0 \\ 2\mathbf{T} \cdot \frac{d\mathbf{T}}{dt} &= 0 \end{aligned}$$

That is, the derivative of $\mathbf{T}$ is orthogonal to $\mathbf{T}$.

We define the *unit normal vector* $\mathbf{N}$ to be

$$\mathbf{N} = \frac{1}{\|d\mathbf{T}/dt\|} \frac{d\mathbf{T}}{dt} \qquad (1)$$

when it exists. Thus, $\mathbf{N}$ is a unit vector which is orthogonal to $\mathbf{T}$



or alternatively, $\mathbf{T}' = \|\mathbf{T}'\| \ \mathbf{N}$.

EXAMPLE 1 Find the unit normal $\mathbf{N}$ to the *helix* $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3t \rangle$

Solution: Since the velocity is $\mathbf{v} = \langle -4\sin(t), 4\cos(t), 3 \rangle$, the speed is

$$v = \sqrt{16\sin^2(t) + 16\cos^2(t) + 9} = \sqrt{9+16} = 5$$

and consequently the unit tangent vector is

$$\mathbf{T} = \frac{1}{v}\mathbf{v} = \left\langle \frac{-4}{5}\sin(t), \frac{4}{5}\cos(t), \frac{3}{5} \right\rangle$$

The derivative of the unit tangent vector is

$$\frac{d\mathbf{T}}{dt} = \frac{d}{dt}\left\langle \frac{-4}{5}\sin(t), \frac{4}{5}\cos(t), \frac{3}{5}\right\rangle = \left\langle \frac{-4}{5}\cos(t), \frac{-4}{5}\sin(t), 0\right\rangle$$

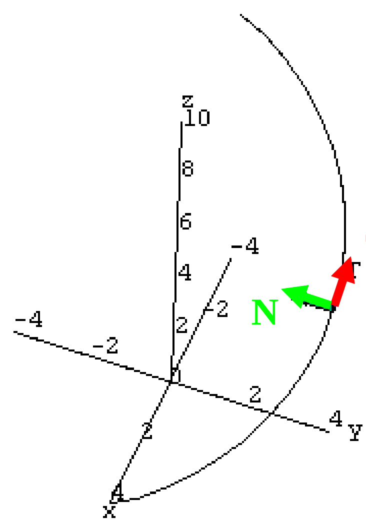
which has a length of

$$\left\|\frac{d\mathbf{T}}{dt}\right\| = \sqrt{\frac{16}{25}\cos^2(t) + \frac{16}{25}\sin^2(t) + 0^2} = \frac{4}{5}$$

Thus, the unit normal is

$$\mathbf{N} = \frac{1}{\|d\mathbf{T}/dt\|}\frac{d\mathbf{T}}{dt} = \frac{1}{4/5}\left\langle \frac{-4}{5}\cos(t), \frac{-4}{5}\sin(t), 0\right\rangle$$

which simplifies to $\mathbf{N} = \langle -\cos(t), -\sin(t), 0\rangle$.



If a curve $\mathbf{r}(s)$ is parameterized by its arclength variable s, then the *curvature* of the curve is defined

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$$

For any other parameterization $\mathbf{r}(t)$, we notice that

$$\left\|\frac{d\mathbf{T}}{dt}\right\| = \left\|\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right\| = \kappa v \tag{2}$$

since $v = ds/dt$. Thus, in general the curvature of a curve is given by

$$\kappa = \frac{1}{v}\left\|\frac{d\mathbf{T}}{dt}\right\|$$

Since $\mathbf{T}' = \|\mathbf{T}'\| \;\mathbf{N}$, equation (2) implies that

$$\frac{d\mathbf{T}}{dt} = \kappa v \;\; \mathbf{N} \tag{3}$$

However, as we will soon see, these formulas only define κ; they are not necessarily the best means of computing κ.

Check your Reading: Is $\mathbf{N}$ the only unit vector orthogonal to $\mathbf{T}$ at a given point on the curve?

The Decomposition of Acceleration

Since a curve's velocity can be written $\mathbf{v} = v\mathbf{T}$ where v is the speed and $\mathbf{T}$ is the unit tangent vector, the acceleration for the curve is

$$\mathbf{a} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{dt} \tag{4}$$

Moreover, if we now combine (4) with (1) and (3) for $d\mathbf{T}/dt$, then we find that the acceleration of a curve parameterized by $\mathbf{r}(t)$ is

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N} \tag{5}$$

The quantity $a_T = dv/dt$ is the rate of change of the speed and is called either the *linear acceleration* or the *tangential component of acceleration* because it measures the acceleration in the direction of the velocity.

The quantity $a_N = \kappa v^2$ is called the *normal component of acceleration* because it measures the acceleration applied at a right angle to the velocity. Specifically, the normal component of acceleration is a measure of how fast the direction of the velocity vector is changing.

Moreover, since $\mathbf{a} \cdot \mathbf{T} = dv/dt$, the decomposition (5) implies that

$$\|\mathbf{a}\|^2 = a_T^2 + a_N^2 = (\mathbf{a} \cdot \mathbf{T})^2 + \kappa^2 v^4$$

Thus, $\kappa^2 v^4 = \|\mathbf{a}\|^2 - (\mathbf{a} \cdot \mathbf{T})^2$, so that

$$\kappa = \frac{\sqrt{\|\mathbf{a}\|^2 - (\mathbf{a} \cdot \mathbf{T})^2}}{v^2} \tag{6}$$

which does not require the calculation of a cross product.

EXAMPLE 2 Find the curvature of the curve parameterized by

$$\mathbf{r}(t) = \langle \sinh(t), t, \cosh(t) \rangle$$

Solution: The velocity is $\mathbf{v}(t) = \langle \cosh(t), 1, \sinh(t) \rangle$, so that the speed is

$$v = \sqrt{\cosh^2(t) + 1 + \sinh^2(t)} = \sqrt{2\cosh^2(t)} = \sqrt{2}\cosh(t)$$

and the unit tangent is

$$\mathbf{T} = \frac{1}{\sqrt{2}\cosh(t)} \langle \cosh(t), 1, \sinh(t) \rangle$$

The derivative of $\mathbf{v}(t)$ then yields the acceleration,

$$\mathbf{a}(t) = \langle \sinh(t), 0, \cosh(t) \rangle$$

and the dot product $\mathbf{a} \cdot \mathbf{T}$ is given by

$$\mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{2}\cosh(t)} (2\sinh(t)\cosh(t)) = \sqrt{2}\sinh(t)$$

Thus, (6) implies that the curvature is

$$\kappa = \frac{\sqrt{\sinh^2(t) + \cosh^2(t) - 2\sinh^2(t)}}{2^2\cosh^2(t)} = \frac{\sqrt{\cosh^2(t) - \sinh^2(t)}}{2^2\cosh^2(t)} = \frac{1}{4\cosh^2(t)}$$

since $\cosh^2(t) - \sinh^2(t) = 1$.

Indeed, if the speed v is constant, then $dv/dt = \mathbf{a} \cdot \mathbf{T} = 0$ and (6) reduces to

$$\kappa = \frac{\sqrt{\|\mathbf{a}\|^2}}{v^2} = \frac{a}{v^2} \tag{7}$$

where a is the magnitude of the acceleration. That is, the curvature of an object moving at a constant speed along a curve is proportional to the magnitude of the acceleration.

EXAMPLE 3 Find the linear acceleration and curvature of the helix

$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$$

Solution: The velocity and acceleration are, respectively, given by

$$\mathbf{v}(t) = \langle -3\sin(t), 3\cos(t), 4 \rangle, \qquad \mathbf{a} = \langle -3\cos(t), -3\sin(t), 0 \rangle$$

It follows that the speed is given by

$$v = \sqrt{9\sin^2(t) + 9\cos^2(t) + 16} = 5$$

Thus, we can use (7). Since $a = 3$, the curvature is thus

$$\kappa = \frac{a}{v^2} = \frac{3}{25}$$

Finally, since $\mathbf{v}$ is parallel to $\mathbf{T}$, the decomposition (5) implies that

$$\mathbf{v} \times \mathbf{a} = \frac{dv}{dt}(\mathbf{v} \times \mathbf{T}) + kv^2(\mathbf{v} \times \mathbf{N}) = \kappa v^2(\mathbf{v} \times \mathbf{N})$$

Since $\mathbf{v}$ and $\mathbf{N}$ are orthogonal, it follows that $\|\mathbf{v} \times \mathbf{N}\| = v \cdot 1 \cdot \sin(\pi/2) = v$, so that

$$\|\mathbf{v} \times \mathbf{a}\| = \kappa v^2 \|\mathbf{v} \times \mathbf{N}\| = \kappa v^3$$

Finally, solving for κ yields another means of computing curvature:

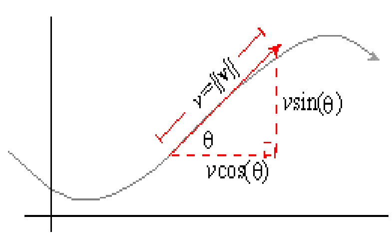
$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{v^3} \tag{8}$$

Check your Reading: Is κ ever 0 in example 5?

Curvature in the Plane

In order to better understand curvature, let's explore it for 2-dimensional curves. If $\mathbf{r}(t)$ is the parameterization of a 2-dimensional curve, then its velocity can be written in polar form as

$$\mathbf{v} = \langle v\cos(\theta), v\sin(\theta)\rangle$$



Factoring out the v then leads to $\mathbf{v} = v\langle\cos(\theta), \sin(\theta)\rangle$, which reveals that the unit vector is

$$\mathbf{T} = \langle\cos[\theta(t)], \sin[\theta(t)]\rangle$$

As a result, the derivative of $\mathbf{T}$ is given by

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= \left\langle \frac{d}{dt}\cos\left[\theta\left(t\right)\right], \frac{d}{dt}\sin\left[\theta\left(t\right)\right] \right\rangle \\ &= \left\langle -\sin\left[\theta\left(t\right)\right]\frac{d\theta}{dt}, \cos\left[\theta\left(t\right)\right]\frac{d\theta}{dt} \right\rangle \\ &= \frac{d\theta}{dt}\left\langle -\sin\left(\theta\right), \cos\left(\theta\right)\right\rangle\end{aligned}$$

Since $\left\langle -\sin\left(\theta\right), \cos\left(\theta\right)\right\rangle$ is also a unit vector, the magnitude of $d\mathbf{T}/dt$ is

$$\left\| \frac{d\mathbf{T}}{dt} \right\| = \frac{d\theta}{dt} \tag{9}$$

which via the chain rule is equivalent to

$$\kappa v = \left\| \frac{d\mathbf{T}}{dt} \right\| = \frac{d\theta}{ds}\frac{ds}{dt} \tag{10}$$

where $s\left(t\right)$ is the arclength function of $\mathbf{r}\left(t\right)$. That is, the curvature of a 2-dimensional curve is given by

$$\kappa = \frac{d\theta}{ds}$$

or equivalently, moving a short distance ds along the curve causes a change $d\theta$ in the direction of the velocity vector, where $d\theta = \kappa ds$.

EXAMPLE 4 Find the curvature of the circle of radius 3 parameterized by

$$\mathbf{r}\left(t\right) = \left\langle 3\cos\left(t\right), 3\sin\left(t\right)\right\rangle$$

Solution: Since the velocity is $\mathbf{v}\left(t\right) = \left\langle -3\sin\left(t\right), 3\cos\left(t\right)\right\rangle$, the speed is

$$v = \sqrt{9\sin^2\left(t\right) + 9\cos^2\left(t\right)} = 3$$

As a result, the unit tangent vector is

$$\mathbf{T}\left(t\right) = \left\langle -\sin\left(t\right), \cos\left(t\right)\right\rangle$$

from which it follows that

$$\frac{d\mathbf{T}}{dt} = \left\langle -\cos\left(t\right), -\sin\left(t\right)\right\rangle$$

As a result, the curvature is

$$\kappa = \frac{1}{v}\left\| \frac{d\mathbf{T}}{dt} \right\| = \frac{1}{3}\sqrt{\cos^2\left(t\right) + \sin^2\left(t\right)} = \frac{1}{3}$$

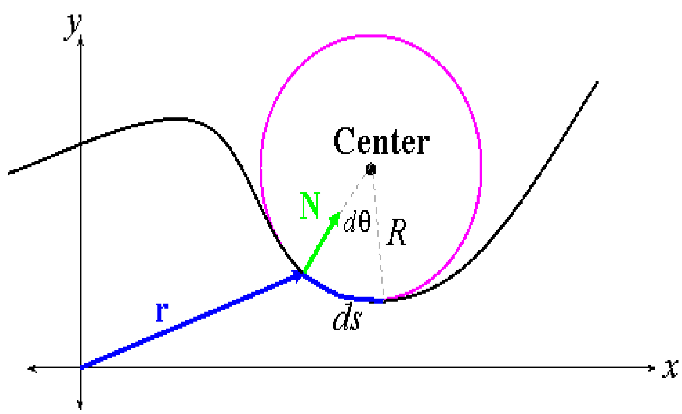
As example 2 illustrates, curvature is closely related to the radius of a circle. Indeed, the *osculating circle* of a curve is the circle with radius

$$R = \frac{1}{\kappa} \quad or \quad equivalently \quad R = \frac{ds}{d\theta}$$

and with center

$$\mathbf{Center}(t) = \mathbf{r}(t) + R\mathbf{N}(t)$$

Consequently, the osculating circle is practically the same as a small section of the curve,



This allows us to interpret $ds = Rd\theta$ to mean that a short distance ds along the curve is practically the same as the small arc with angle $d\theta$ on the osculating circle.

Moreover, since working the osculating circle requires the unit normal vector $\mathbf{N}$, we often use

$$\kappa = \frac{1}{v}\frac{d\theta}{dt} = \frac{1}{v}\left\|\frac{d\mathbf{T}}{dt}\right\|$$

to calculate the curvature (since we must calculate $\|d\mathbf{T}/dt\|$ already in order to obtain $\mathbf{N}$).

EXAMPLE 5 What is the curvature of the curve $\mathbf{r}(t) = \langle \ln|\cos t|, t\rangle$ for t in $[-\pi/3, \pi/2]$.

Solution: The velocity is

$$\mathbf{v} = \left\langle \frac{1}{\cos(t)}\frac{d}{dt}\cos(t), \quad 1\right\rangle = \left\langle \frac{-\sin(t)}{\cos(t)}, 1\right\rangle$$

which reduces to $\mathbf{v} = \langle -\tan(t), 1 \rangle$. Thus, the speed is

$$v = \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)} = \sec(t)$$

since $\sec(t) > 0$ for t in $[-\pi/3, \pi/2]$. The unit tangent is consequently

$$\mathbf{T} = \frac{1}{\sec(t)} \langle -\tan(t), 1 \rangle = \langle -\sin(t), \cos(t) \rangle$$

and the derivative of the unit tangent is

$$\frac{d\mathbf{T}}{dt} = \langle -\cos(t), -\sin(t) \rangle$$

(which is also the unit normal vector $\mathbf{N}$). As a result, (**??**) implies that the curvature is

$$\kappa = \frac{1}{v} \left\| \frac{d\mathbf{T}}{dt} \right\| = \frac{1}{\sec(t)} \sqrt{\cos^2(t) + \sin^2(t)} = \cos(t)$$

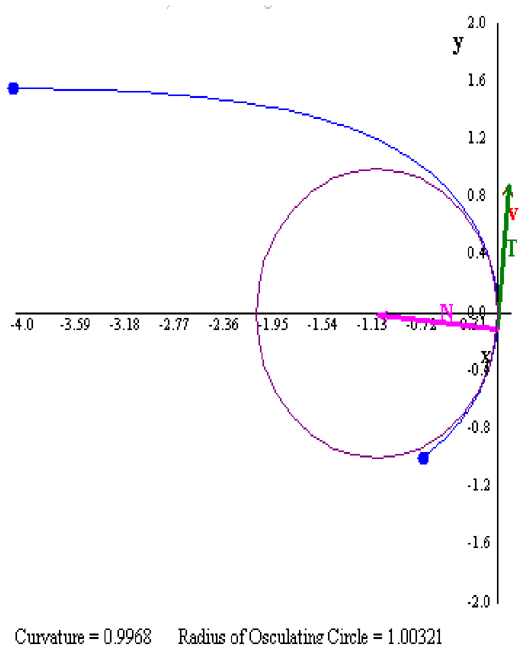
It follows that $R = 1/\kappa$ and that

$$\begin{aligned} \mathbf{Center}(t) &= \mathbf{r}(t) + R\mathbf{N}(t) \\ &= \langle \ln|\cos t|, t \rangle + \frac{1}{\cos(t)} \langle -\cos(t), -\sin(t) \rangle \\ &= \langle \ln|\cos t| - 1, t - \tan(t) \rangle \end{aligned}$$

Indeed, the osculating circle itself at a fixed time t is given by

$$\begin{aligned} \mathbf{Osc}(\theta) &= \mathbf{Center}(t) + R \langle \cos(\theta), \sin(\theta) \rangle \\ &= \langle \ln|\cos t| - 1, t - \tan(t) \rangle + \sec(t) \langle \cos(\theta), \sin(\theta) \rangle \end{aligned}$$

This is illustrated in the applet below:



If $\mathbf{r}(t)$ parameterizes a straight line, then its unit tangent is constant, and correspondingly, its curvature is $\kappa = 0$. To illustrate, the curve $\mathbf{r}(t) = \langle \ln |\cos(t)|, 1 \rangle$ in example 3 approaches a *horizontal asymptote* (i.e., a straight line) as t approaches $\pi/2$, and likewise, the curvature approaches 0 as t approaches $\pi/2$. Since the relationship of the curvature and the radius of the osculating circle is

$$\kappa = \frac{1}{\text{radius of the osculating circle}}$$

the osculating circle becomes infinitely large as κ approaches 0.

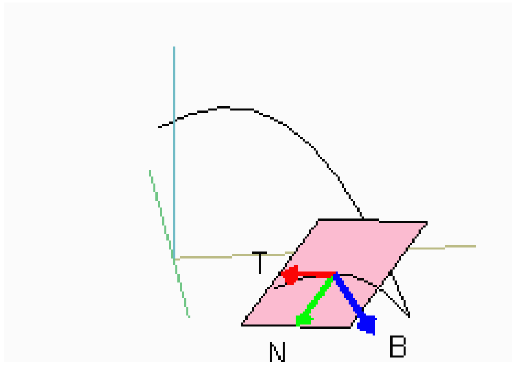
Check your Reading: How is the curvature in example 2 related to the radius of the circle?

Torsion and the Frenet Frame

Given a curve $\mathbf{r}(t)$ in space, the *binormal vector* $\mathbf{B}$ is defined

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Thus, $\mathbf{B}$ is a unit vector normal to the plane spanned by $\mathbf{T}$ and $\mathbf{N}$ at time t.



Since both $\mathbf{v}$ and $\mathbf{a}$ are in the plane spanned by $\mathbf{T}$ and $\mathbf{N}$, the binormal vector $\mathbf{B}$ is also the unit vector in the direction of $\mathbf{v} \times \mathbf{a}$. That is,

$$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{\|\mathbf{v} \times \mathbf{a}\|}$$

Moreover, if $\mathbf{B}$ is constant, then the curve $\mathbf{r}(t)$ must be contained in a plane with normal $\mathbf{B}$.

EXAMPLE 6 Find $\mathbf{B}$ for $\mathbf{r}(t) = \langle \sin(t), \cos(t), \sin(t) \rangle$. Is $\mathbf{r}(t)$ in a plane?

Solution: Since $\mathbf{v}(t) = \langle \cos(t), -\sin(t), \cos(t) \rangle$ and $\mathbf{a}(t) = \langle -\sin(t), -\cos(t), -\sin(t) \rangle$, their cross product is

$$\mathbf{v} \times \mathbf{a} = \left\langle \begin{vmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{vmatrix}, \begin{vmatrix} \cos(t) & \cos(t) \\ -\sin(t) & -\sin(t) \end{vmatrix}, \begin{vmatrix} \cos(t) & -\sin(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} \right\rangle$$

which simplifies to

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \left\langle \sin^2(t) + \cos^2(t), 0, -\cos^2(t) - \sin^2(t) \right\rangle \\ &= \langle 1, 0, -1 \rangle \end{aligned}$$

Since $\|\mathbf{v} \times \mathbf{a}\| = \sqrt{2}$, the unit binormal is

$$\mathbf{B} = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$$

Moreover, $\mathbf{B}$ is constant, so $\mathbf{r}(t)$ is confined to a single plane.

Finally, the definition $\mathbf{B}=\mathbf{T}\times\mathbf{N}$ implies that

$$\frac{d\mathbf{B}}{dt}=\frac{d\mathbf{T}}{dt}\times\mathbf{N}+\mathbf{T}\times\frac{d\mathbf{N}}{dt}$$

However, $d\mathbf{T}/dt$ is parallel to $\mathbf{N}$, so that $d\mathbf{T}/dt\times\mathbf{N}=0$ and

$$\frac{d\mathbf{B}}{dt}=\mathbf{T}\times\frac{d\mathbf{N}}{dt}$$

Thus, $d\mathbf{B}/dt$ must be perpendicular to $\mathbf{T}$. Moreover, the fact that $\mathbf{B}(t)$ is a unit vector for all t implies that $d\mathbf{B}/dt$ is also perpendicular to $\mathbf{B}$. Thus, $d\mathbf{B}/dt$ is parallel to $\mathbf{N}$, which means that

$$\frac{d\mathbf{B}}{dt}=-\tau\ v\mathbf{N} \tag{11}$$

where the constant of proportionality τ is known as the *torsion* of the curve. It follows that

$$\tau=-\frac{1}{v}\mathbf{N}\cdot\frac{d\mathbf{B}}{dt}$$

Torsion is a measure of how much the plane spanned by $\mathbf{T}$ and $\mathbf{N}$ "osculates" as the parameter increases. For example, if $\mathbf{r}(t)$ is a curve contained in a single fixed plane, then $\mathbf{B}$ must be constant and consequently, the torsion $\tau=0$. In fact, $\tau=0$ only if motion is in a plane. More properties of τ will be explored in the exercises. Torsion will be explored in more detail in the exercises.

Exercises

Find the unit normal $\mathbf{N}$ and the curvature $\kappa(t)$ of each of the following curves:

1. $\mathbf{r}(t)=\langle 3t,4t+3\rangle$
2. $\mathbf{r}(t)=\langle 5t+2,12t+3\rangle$
3. $\mathbf{r}(t)=\langle\cos(2t),\sin(2t)\rangle$
4. $\mathbf{r}(t)=\langle 3\cos(\pi t),3\sin(\pi t)\rangle$
5. $\mathbf{r}(t)=\langle 3\sin(t),3\cos(t),4t\rangle$
6. $\mathbf{r}(t)=\left\langle t^3,3t^2,6t\right\rangle$
9. $\mathbf{r}(t)=\left\langle 3\sin\left(t^2\right),4\sin\left(t^2\right),5\cos\left(t^2\right)\right\rangle$
10. $\mathbf{r}(t)=\left\langle e^{2t},2e^t,t\right\rangle$

Find the linear acceleration dv/dt and the curvature $\kappa(t)$ of each of the following curves:

11. $\mathbf{r}(t)=\langle\cos(2t),\sin(2t)\rangle$
12. $\mathbf{r}(t)=\langle 3\cos(\pi t),3\sin(\pi t)\rangle$
13. $\mathbf{r}(t)=\left\langle t,2t^{3/2},2(1-t)^{3/2}\right\rangle$
14. $\mathbf{r}(t)=\langle 2t,3t,4t+1\rangle$
15. $\mathbf{r}(t)=\left\langle 2\cos\left(t^{3/2}\right),2\sin\left(t^{3/2}\right),2(10-t)^{3/2}\right\rangle$
16. $\mathbf{r}(t)=\left\langle\cos\left(t^{3/2}\right),\sin\left(t^{3/2}\right),(4-t)^{3/2}\right\rangle$
17. $\mathbf{r}(t)=\langle 3t+4\sin(t),4t-3\sin(t),5\cos(t)\rangle$
18. $\mathbf{r}(t)=\left\langle t+\sin(t),t-\sin(t),\sqrt{2}\cos(t)\right\rangle$
19. $\mathbf{r}(t)=\langle\sin(t),\cosh(t),\cos(t)\rangle$
20. $\mathbf{r}(t)=\left\langle\sin^2(t),\sin^2(t),\cos(2t)\right\rangle$
21. $\mathbf{r}(t)=\left\langle e^{2t},2e^t,t\right\rangle$
22. $\mathbf{r}(t)=\left\langle t^2,2t,\ln(t)\right\rangle$

Find the unit binormal and the torsion of each curve. Is the curve restricted to a plane?

23.	$\mathbf{r}(t) = \langle t, t, t^2 \rangle$	24.	$\mathbf{r}(t) = \langle 2t, 3t, 4t+1 \rangle$
25.	$\mathbf{r}(t) = \langle 3\sin(t), 3\cos(t), 4t \rangle$	26.	$\mathbf{r}(t) = \langle 3\sin(t^2), 3\cos(t^2), 4t^2 \rangle$
27.	$\mathbf{r}(t) = \langle 3\sin(t), 5\cos(t), 4\sin(t) \rangle$	28.	$\mathbf{r}(t) = \langle \sin(t), \cosh(t), \cos(t) \rangle$

29. Find the equation of the line between the points $P_1(1,2,1)$ and $P_2(2,3,1)$. Then find its linear acceleration dv/dt and its curvature. What is the curvature of a straight line and why?

30. Show that the graph of the vector-valued function

$$\mathbf{r}(t) = \langle \sec^2(t), \tan^2(t) \rangle$$

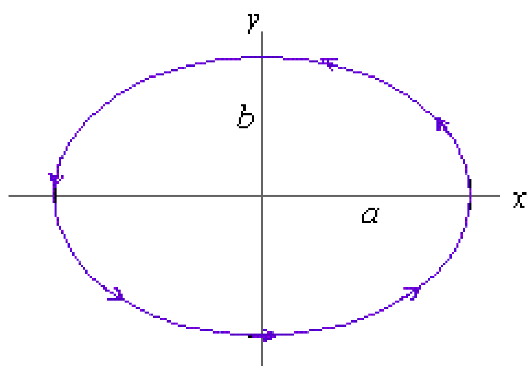
is a straight line. Then find its acceleration and its curvature.

31. Show that the graph of the vector-valued function

$$\mathbf{r}(t) = \langle 4\cos^2(t), 2\sin(2t) \rangle$$

is a circle by showing that it has constant curvature. (Hint: $4\cos^2(2t) - 2 = 2(2\cos^2(t) - 1)$)

32. An ellipse with semi-major axis a and semi-minor axis b



can be parameterized by

$$\mathbf{r}(t) = \langle a\cos(t), b\sin(t) \rangle$$

for t in $[0, 2\pi]$. Show that the curvature of the ellipse is

$$\kappa(t) = \frac{ab}{\left(a^2\sin^2(t) + b^2\cos^2(t)\right)^{3/2}}$$

What is the curvature of the ellipse when $a = b$?

33. The function $\mathbf{r}(t) = \langle t, t^3 \rangle$ parameterizes the curve $y = x^3$. Find the curvature of the curve and determine where it is equal to 0. What is significant about this point on the curve?

34. Show that any curve with zero curvature must also have zero torsion.
35. Show that if $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then the curvature at time t is given by

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

36. Use the fact that $\mathbf{r}(t) = \langle t, f(t) \rangle$ parameterizes the curve $y = f(x)$ to show that the curvature of the graph of a second differentiable function $f(x)$ is

$$\kappa(x) = \frac{|f''|}{\left[1 + (f')^2\right]^{3/2}}$$

37. Explain in your own words why at any point on a 3-dimensional smooth curve, the osculating circle must be in the plane with $\mathbf{B}$ as a normal.
38. The curve $\mathbf{r}(t) = \langle R\cos(t), R\sin(t) \rangle$ is a circle centered at the origin. Compute the center of its osculating circle. Is it what you expected?
39. Use the triple vector product to prove that

$$\mathbf{T} = \mathbf{N} \times \mathbf{B} \qquad \textit{and} \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T}$$

and then use the result to show that

$$\tau = \frac{1}{v}\mathbf{B} \cdot \frac{d\mathbf{N}}{dt}$$

40. The general form of a helix which spirals about the z-axis is given by

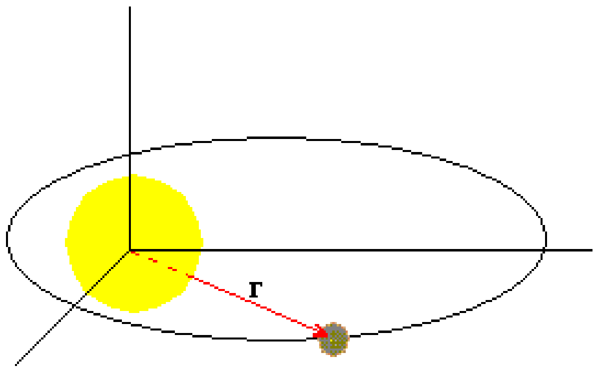
$$\mathbf{r}(t) = \langle a\cos(t), a\sin(t), bt \rangle$$

where $a > 0$ and $b > 0$. Compute the curvature κ and torsion τ of the helix. How are they related to a and b.
41. In this problem, we consider the "compressed helix"

$$\mathbf{r}(t) = \left\langle \cos(t), \sin(t), e^{-t} \right\rangle$$

1. (a) What happens to the helix as t approaches ∞?
   (b) What value does $\kappa(t)$ approach as t approaches ∞?
   (c) What value does $\tau(t)$ approach as t approaches ∞?

42. Suppose that $\mathbf{r}(t)$ is the position at time t of a planet as it orbits a sun located at the origin of a 3-dimensional coordinate system.



The angular velocity of the planet is $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ and the acceleration of the planet is

$$\mathbf{a} = \frac{-GM}{r^3}\mathbf{r}$$

where M is the mass of the sun and G is the universal gravitational constant. Show that

$$\mathbf{v} \times \mathbf{a} = \frac{GM}{r^3}\mathbf{L}$$

and then use this result to express the curvature of the planet's orbit as a function of r, v, G, M, and L.

43. Show that if $\mathbf{r}(s)$ is parameterized by the arclength variable (that is, $v = 1$), then

$$\mathbf{v} = \mathbf{T}, \quad \mathbf{a} = \mathbf{N}, \quad \mathbf{B} = \mathbf{v} \times \mathbf{a}$$

and that

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| \qquad and \qquad \tau = \frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

44. Write to Learn: Write a short essay in which you show that a curve $\mathbf{r}(t)$ has zero torsion (i.e., $\tau = 0$) if and only if $\mathbf{r}(t)$ is a motion in a fixed plane.

45. Write to Learn: Write a short essay discussing the relationship between an automobile's odometer, speedometer, and accelerator. Does any instrument in an automobile measure the curvature of the automobile's path? Or are all the instruments and controls in an automobile related strictly to the linear components of acceleration?