The Cross Product

Part 1: Determinants and the Cross Product

In this section, we introduce the *cross product* of two vectors. However, the cross product is based on the *theory of determinants*, so we begin with a review of the properties of determinants.

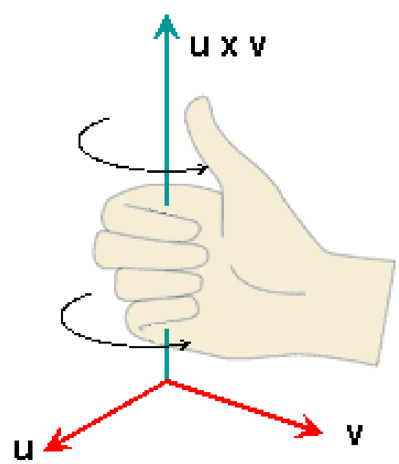
To begin with, the *determinant* of a 2×2 array of numbers is defined

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1 \tag{1}$$

Subsequently, the *cross product* of the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined to be a vector of determinants:

$$\mathbf{u} \times \mathbf{v} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle \tag{2}$$

We have constructed (2) so that the direction of $\mathbf{u} \times \mathbf{v}$ satisfies the *right-hand rule,* which says that as the fingers of the right hand sweep from $\mathbf{u}$ to $\mathbf{v}$ through an angle of less than $180°$, the thumb points in the direction of $\mathbf{u} \times \mathbf{v}$



EXAMPLE 1 Compute $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ for $\mathbf{u} = \langle 2, 3, 5 \rangle$ and $\mathbf{v} = \langle 6, 7, 9 \rangle$.

Solution: To do so, we construct the vector of determinants in (2),

$$\mathbf{u} \times \mathbf{v} = \left\langle \begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 9 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} \right\rangle$$

and then we use (1) to evaluate the determinants:

$$\mathbf{u} \times \mathbf{v} = \langle 3 \cdot 9 - 7 \cdot 5, 5 \cdot 6 - 9 \cdot 2, 2 \cdot 7 - 6 \cdot 3 \rangle = \langle -8, 12, -4 \rangle \quad (3)$$

Notice however that $\mathbf{v} \times \mathbf{u}$ is

$$\mathbf{v} \times \mathbf{u} = \left\langle \begin{vmatrix} 7 & 9 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} 9 & 6 \\ 5 & 2 \end{vmatrix}, \begin{vmatrix} 6 & 7 \\ 2 & 3 \end{vmatrix} \right\rangle = \langle 8, -12, 4 \rangle$$

That is, $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$, which can be shown to be true in general. Indeed, each of the following follow from direct calculation.

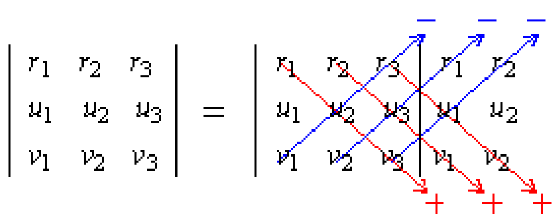
Theorem 3.1: The cross product is defined *only* for 3 dimensional vectors $\mathbf{u}$ and $\mathbf{v}$. Moreover, the following hold:

i) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
ii) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
iii) $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times k\mathbf{v}$
iv) $\mathbf{a} \times (\mathbf{u} + \mathbf{v}) = \mathbf{a} \times \mathbf{u} + \mathbf{a} \times \mathbf{v}$

The determinant of a 3×3 array of numbers is defined

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = r_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + r_2 \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + r_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \quad (4)$$

Alternatively, we can calculate (4) by repeating the first two columns after the determinant, and then computing the sums of the products along the 3 main diagonals and the 3 off diagonals. The determinant is then the difference of the two sums:



(5)

If we put $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in the first row of (4), we obtain

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Thus, we can also calculate $\mathbf{u} \times \mathbf{v}$ using a 3-dimensional determinant.

EXAMPLE 2 Use a 3-dimensional determinant to compute $\mathbf{u} \times \mathbf{v}$ when $\mathbf{u} = \langle 2, 1, 2 \rangle$ and $\mathbf{v} = \langle 3, 4, 5 \rangle$

Solution: We write $\mathbf{u} \times \mathbf{v}$ as a 3 dimensional determinant and then evaluate the determinant.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 2 & 1 & 2 & 2 & 1 \\ 3 & 4 & 5 & 3 & 4 \end{array} \right| = 5\mathbf{i} + 6\mathbf{j} + 8\mathbf{k} - 3\mathbf{k} - 8\mathbf{i} - 10\mathbf{j}$$

This then simplifies to $\mathbf{u} \times \mathbf{v} = -3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

Check Your Reading: What is $\mathbf{v} \times \mathbf{u}$ when $\mathbf{u} = \langle 2, 1, 2 \rangle$ and $\mathbf{v} = \langle 3, 4, 5 \rangle$?

The Triple Scalar Product

If $\mathbf{r} = \langle r_1, r_2, r_3 \rangle$, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then (4) implies that

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

However, by (5) we have

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = \left| \begin{array}{ccc|cc} r_1 & r_2 & r_3 & r_1 & r_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array} \right| = \left| \begin{array}{ccc|cc} v_1 & v_2 & v_3 & v_1 & v_2 \\ r_1 & r_2 & r_3 & r_1 & r_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \end{array} \right| = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}$$

We call this identity the *triple scalar product:*

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}$$

The triple scalar product is often used to obtain other properties of the cross product. For example, notice that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v} = 0$$

since $\mathbf{u} \times \mathbf{u} = 0$. Since $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u})$, the same calculation shows that that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$.

Theorem 3.2: $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and to $\mathbf{v}$.

This property of the cross product is important in many applications.

EXAMPLE 3 Show that $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ when $\mathbf{u}=\langle 2,3,7\rangle$ and $\mathbf{v}=\langle 1,4,2\rangle$.

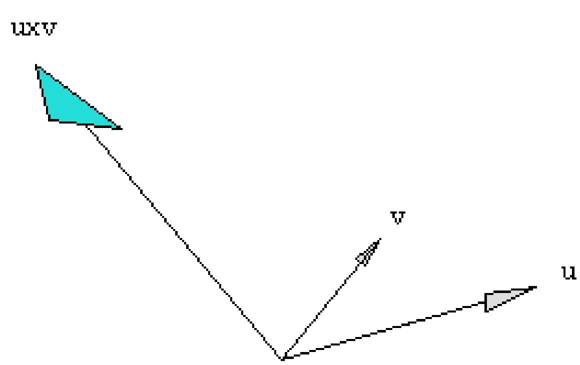
Solution: To begin with, let us notice that

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \left\langle \begin{vmatrix} 3 & 7 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} 7 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \right\rangle \\ &= \langle 6-28, 7-4, 8-3\rangle \\ &= \langle -22, 3, 5\rangle\end{aligned}$$

Now let's compute $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})$:

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = \langle 2,3,7\rangle\cdot\langle -22,3,5\rangle = -44+9+35=0$$

and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ are shown below:



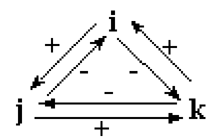
As another example, consider that

$$\mathbf{i}\times\mathbf{j} = \left\langle \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right\rangle = \langle 0,0,1\rangle$$

That is, $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, which is perpendicular to both $\mathbf{i}$ and $\mathbf{j}$. Indeed, it can be shown that

$$\begin{array}{lll} \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{i}\times\mathbf{i}=\mathbf{0} \\ \mathbf{j}\times\mathbf{k}=\mathbf{i} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{j}\times\mathbf{j}=\mathbf{0} \\ \mathbf{k}\times\mathbf{i}=\mathbf{j} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} & \mathbf{k}\times\mathbf{k}=\mathbf{0} \end{array} \tag{6}$$

The identities (6) can be memorized using the mnemonic below:



That is, the cross product of any two vertices is the third vertex, with the sign determined by the direction implied by the order of the first two vertices in reaching the third.

EXAMPLE 4 Evaluate $\mathbf{u} \times \mathbf{v}$ when $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = \mathbf{j} + \mathbf{k}$. Then show that $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{u}$ and $(\mathbf{u} \times \mathbf{v}) \perp \mathbf{v}$

Solution: To begin with, we use property iii):

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (\mathbf{i} - 2\mathbf{j}) \times (\mathbf{j} + \mathbf{k}) \\ &= \mathbf{i} \times (\mathbf{j} + \mathbf{k}) - 2\mathbf{j} \times (\mathbf{j} + \mathbf{k}) \\ &= \mathbf{i} \times \mathbf{j} + \mathbf{i} \times \mathbf{k} - 2\mathbf{j} \times \mathbf{j} - 2\mathbf{j} \times \mathbf{k} \end{aligned}$$

We then use table (6) to finish the computation:

$$\mathbf{u} \times \mathbf{v} = \mathbf{k} - \mathbf{j} - 2 \cdot 0 - 2\mathbf{i} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

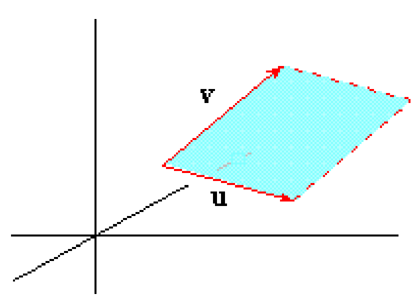
Notice that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1\,(-2) - 2\,(-1) = 0$ and

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1\,(-1) - 1\,(1) = 0$$

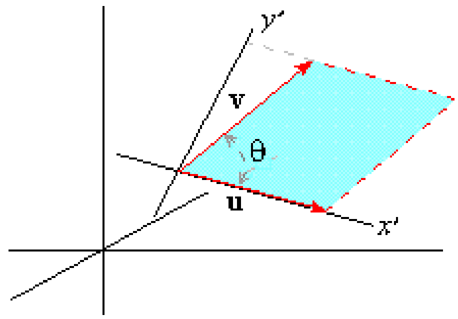
Check your Reading: What is $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$?

Parallelograms and Triangles

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ with a common initial point, the set of terminal points of the vectors $s\mathbf{u} + t\mathbf{v}$ for $0 \leq s, t \leq 1$ is defined to be *parallelogram spanned by* $\mathbf{u}$ and $\mathbf{v}$.



We can explore the parallelogram spanned by two vectors in a 2-dimensional coordinate system. That is, because coordinate systems are a figment of our collective imaginations, we can imagine the parallelogram spanned by two vectors as being in an $x'y'$ coordinate system, where the x'-axis is parallel to $\mathbf{u}$ and the y'-axis is in the same plane as $\mathbf{u}$ and $\mathbf{v}$.



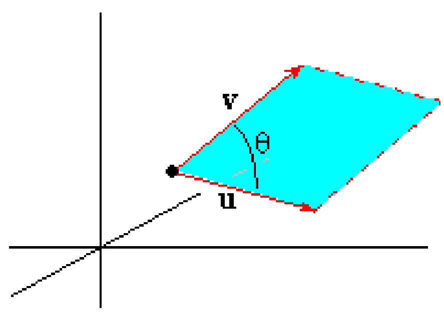
Since $\mathbf{u} = ||\mathbf{u}||\,\mathbf{i}$ and $\mathbf{v} = ||\mathbf{v}||\,(\cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j})$ in the $x'y'$ coordinate system, their cross product is

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= ||\mathbf{u}||\ \ ||\mathbf{v}||\ \ (\cos(\theta)\,\mathbf{i}\times\mathbf{i} + \sin(\theta)\,\mathbf{i}\times\mathbf{j}) \\ &= ||\mathbf{u}||\ \ ||\mathbf{v}||\ \ \sin(\theta)\,\mathbf{k}\end{aligned}$$

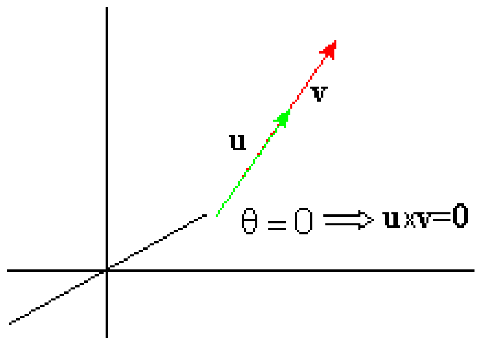
This results in the following theorem:

Theorem 3.3: If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

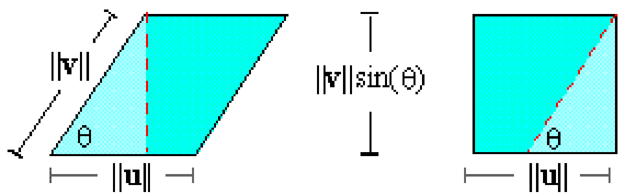
$$||\mathbf{u}\times\mathbf{v}|| = ||\mathbf{u}||\ \ ||\mathbf{v}||\ \ \sin(\theta)$$



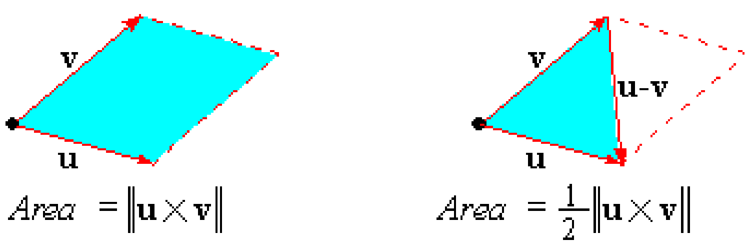
Let's consider two applications of Theorem 3.3. First, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\theta = 0$ and $\mathbf{u}\times\mathbf{v} = 0$. s=



Second, the parallelogram spanned by **u** and **v** can be cut into two parts which form a rectangle with height $\|\mathbf{v}\|\sin(\theta)$ and base $\|\mathbf{u}\|$,



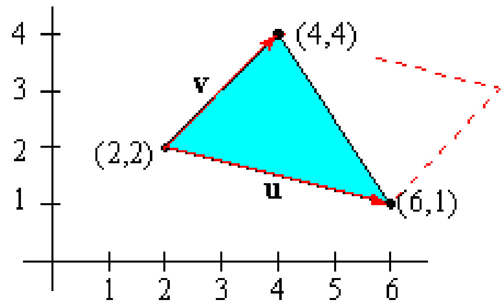
Thus, the area of the parallelogram spanned by **u** and **v** is $\|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\theta)$. Indeed, we have the following:



The latter result follows from the fact that $\mathbf{u}-\mathbf{v}$ bisects the parallelogram formed by **u** and **v**.

EXAMPLE 5 Find the area of the triangle with vertices at $P_1(2,2)$,

$P_2\,(4,4)$, and $P_3\,(6,1)$.



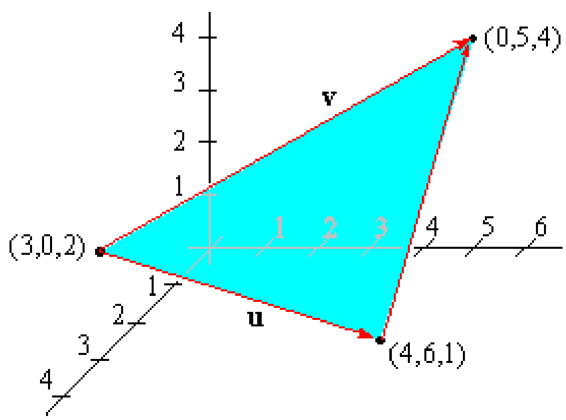
Solution: It is easy to see that $\mathbf{u} = \langle 2,2\rangle$ and $\mathbf{v} = \langle 4,-1\rangle$. As vectors in $\mathbb{R}^3$, we have $\mathbf{u} = \langle 2,2,0\rangle$ and $\mathbf{v} = \langle 4,-1,0\rangle$. Thus, their cross product is

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \left\langle \begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 4 & -1 \end{vmatrix} \right\rangle \\
&= \langle 0, 0,\ 2\cdot(-1) - 4\cdot 2\ \rangle \\
&= \langle 0, 0, -10\rangle
\end{aligned}$$

Since the triangle has half of the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, the area of the triangle is

$$Area = \|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\sqrt{0^2 + 0^2 + (-10)^2} = 5\ units^2$$

EXAMPLE 6 Find the area of the triangle with vertices $P_1\,(3,0,2)$, $P_2\,(4,6,1)$, and $P_3\,(0,5,4)$.



Solution: To do so, we first construct the vectors $\mathbf{u}$ and $\mathbf{v}$:

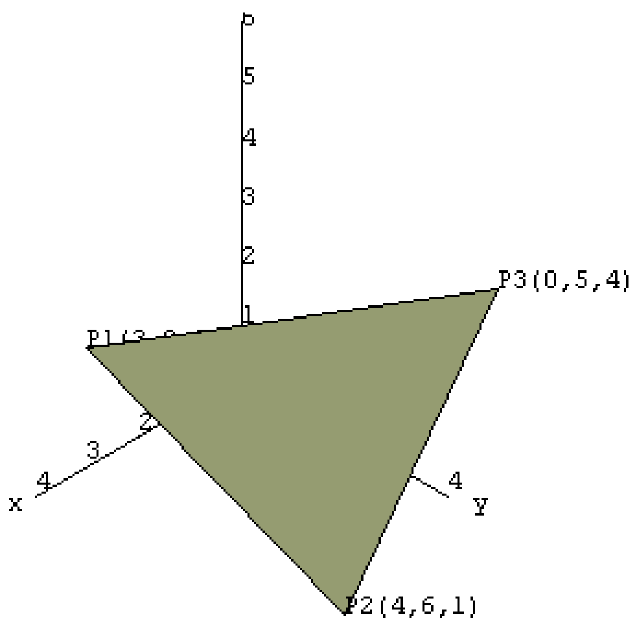
$$\begin{aligned}
\mathbf{u} &= \overrightarrow{P_1P_2} = \langle 4-3, 6-0, 1-2\rangle = \langle 1, 6, -1\rangle \\
\mathbf{v} &= \overrightarrow{P_1P_3} = \langle 0-3, 5-0, 4-2\rangle = \langle -3, 5, 2\rangle
\end{aligned}$$

As vectors in $\mathbb{R}^3$, we now have $\mathbf{u} = \langle 2, 2, 0 \rangle$ and $\mathbf{v} = \langle 4, -1, 0 \rangle$. Thus, their cross product is

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \left\langle \begin{vmatrix} 6 & -1 \\ 5 & 2 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}, \begin{vmatrix} 1 & 6 \\ -3 & 5 \end{vmatrix} \right\rangle \\
&= \langle 6 \cdot 2 - 5 \cdot (-1), (-1) \cdot (-3) - 2 \cdot 1, 1 \cdot 5 - (-3) \cdot 6 \rangle \\
&= \langle 17, 1, 23 \rangle
\end{aligned}$$

Since the triangle has half of the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, the area of the triangle is

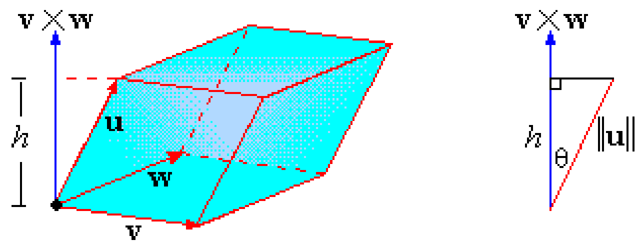
$$Area = \|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{17^2 + 2^2 + 23^2} = 14.335 \; units^2$$



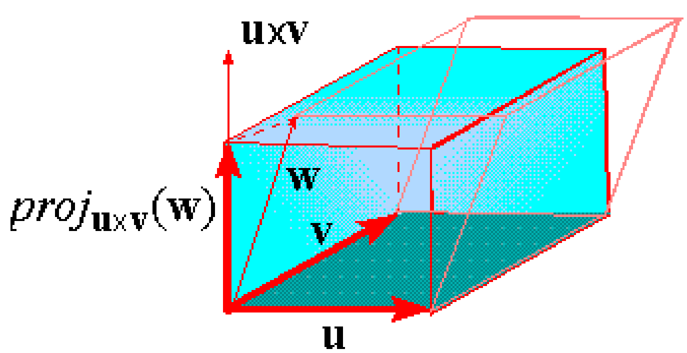
Check your Reading: What is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ in example 4?

Volume of a Parallelpiped

If $\mathbf{u}, \mathbf{v}$,and $\mathbf{w}$ share a common initial point, then the set of terminal points of the vectors $s\mathbf{u} + t\mathbf{v} + r\mathbf{w}$ for $0 \leq r, s, t \leq 1$ is called the *parallelpiped spanned by* $\mathbf{u}, \mathbf{v}, and\ \mathbf{w}$:



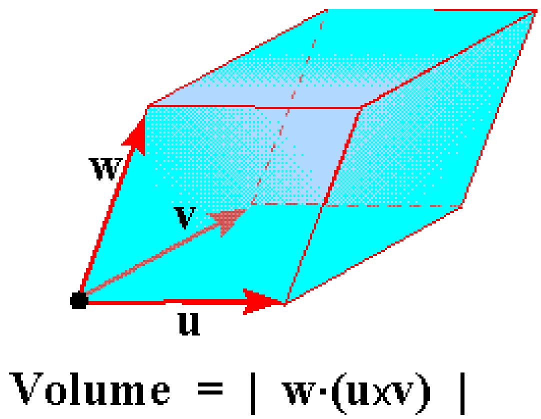
In order to find the volume of the parallelpiped, we first notice that the parallelpiped can be "sliced" into parts that can be rearranged to form a new parallelpiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $proj_{\mathbf{u}\times\mathbf{v}}(\mathbf{w})$. **Click the slideshow arrows below the image to see this slicing and rearranging in action.**



The new parallelpiped and the old parallelpiped have the same volume, and because $proj_{\mathbf{u}\times\mathbf{v}}(\mathbf{w})$ is perpendicular to the base spanned by $\mathbf{u}$ and $\mathbf{v}$, the volume of the new parallelpiped is

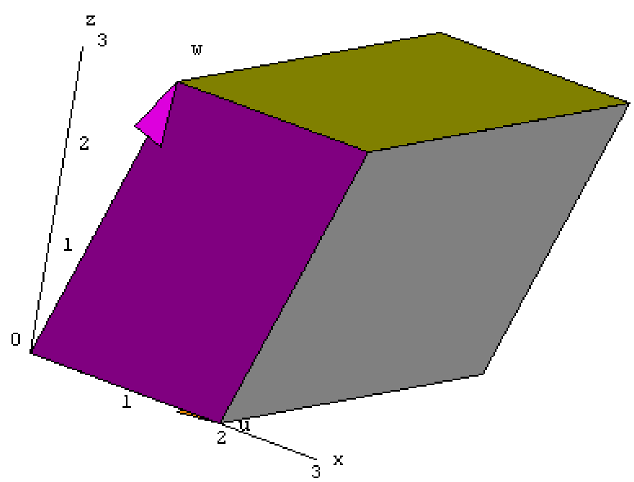
$$\begin{aligned}
Volume &= (Area\ of\ base)\,(height) \\
&= \|\mathbf{u}\times\mathbf{v}\|\,\|proj_{\mathbf{u}\times\mathbf{v}}(\mathbf{w})\| \\
&= \|\mathbf{u}\times\mathbf{v}\|\left\|\frac{\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})}{(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v})}(\mathbf{u}\times\mathbf{v})\right\| \\
&= \|\mathbf{u}\times\mathbf{v}\|\frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{\|\mathbf{u}\times\mathbf{v}\|^2}\|\mathbf{u}\times\mathbf{v}\| \\
&= |\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|
\end{aligned}$$

Thus, the volume of the parallelpiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is



which is known as the *triple scalar product.* Equivalently, $Volume = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$

EXAMPLE 7 Find the volume of the parallelpiped spanned by $\mathbf{u} = \langle 2, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 3, 0 \rangle$, and $\mathbf{w} = \langle 1, 0, 3 \rangle$. The figure below is drawn as if all vectors have their initial points at the origin.



Solution: The cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \left\langle \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} \right\rangle = \langle 0, 0, 6 \rangle$$

The dot product with $\mathbf{w}$ yields the volume:

$$Volume = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\langle 1, 0, 3\rangle \cdot \langle 0, 0, 6\rangle| = 18$$

Exercises

Compute the cross product of $\mathbf{u} \times \mathbf{v}$ *and then compute the cross product of* $\mathbf{v} \times \mathbf{u}$. *Also, show that* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal to* $\mathbf{u} \times \mathbf{v}$.

1. $\mathbf{u} = \langle 2, 1, 0\rangle$, $\mathbf{v} = \langle 3, 1, 0\rangle$
2. $\mathbf{u} = \langle 2, 1, 0\rangle$, $\mathbf{v} = \langle -1, 3, 0\rangle$
3. $\mathbf{u} = \langle 3, 3, 0\rangle$, $\mathbf{v} = \langle 2, 0, 0\rangle$
4. $\mathbf{u} = \langle 0, 1, 0\rangle$, $\mathbf{v} = \langle 0, 0, 1\rangle$
5. $\mathbf{u} = \langle 1, 0, 0\rangle$, $\mathbf{v} = \langle 0, 1, 0\rangle$
6. $\mathbf{u} = \langle 1, 0, 0\rangle$, $\mathbf{v} = \langle 1, 0, 0\rangle$
7. $\mathbf{u} = \langle 2, 3, 7\rangle$, $\mathbf{v} = \langle 7, 3, 5\rangle$
8. $\mathbf{u} = \langle 6, 2, 9\rangle$, $\mathbf{v} = \langle 1, 0, 3\rangle$
9. $\mathbf{u} = \langle 3, 4, 2\rangle$, $\mathbf{v} = \langle 9, 12, 6\rangle$
10. $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle -1, -1, -1\rangle$

Sketch the triangle formed by the three points P_1, P_2 *and* P_3 *and then find its area.*

11. $P_1(0, 0)$, $P_2(1, 2)$, $P_3(2, 1)$
12. $P_1(0, 0)$, $P_2(2, 3)$, $P_3(1, 1)$
13. $P_1(1, 3)$, $P_2(-2, 5)$, $P_3(2, 1)$
14. $P_1(-1, 4)$, $P_2(3, 4)$, $P_3(0, -3)$
15. $P_1(1, 0, 0)$, $P_2(0, 1, 0)$, $P_3(0, 0, 1)$
16. $P_1(0, 0, 0)$, $P_2(1, 1, 0)$, $P_3(1, 1, 1)$
17. $P_1(0, 0, 0)$, $P_2(1, 2, 1)$, $P_3(2, 1, 2)$
18. $P_1(1, 3, 2)$, $P_2(2, 7, 9)$, $P_3(2, 1, 5)$

Sketch the parallelpiped spanned by $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *assuming all vectors have initial points at the origin. Then use the triple scalar product to calculate the volume of the parallelpiped.*

19. $\mathbf{u} = \langle 2, 0, 0\rangle$, $\mathbf{v} = \langle 1, 2, 0\rangle$, $\mathbf{w} = \langle 0, 0, 3\rangle$
20. $\mathbf{u} = \langle 1, 0, 0\rangle$, $\mathbf{v} = \langle 1, 1, 0\rangle$, $\mathbf{w} = \langle 1, 1, 1\rangle$
21. $\mathbf{u} = \langle 2, 0, 0\rangle$, $\mathbf{v} = \langle 0, 2, 0\rangle$, $\mathbf{w} = \langle 1, 1, 1\rangle$
22. $\mathbf{u} = \langle -1, 2, 0\rangle$, $\mathbf{v} = \langle 2, 1, 0\rangle$, $\mathbf{w} = \langle 1, 3, 2\rangle$
23. $\mathbf{u} = \mathbf{i} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$, $\mathbf{w} = \mathbf{j} + 2\mathbf{k}$
24. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{k} + \mathbf{i}$
25. $\mathbf{u} = \langle l, 0, 0\rangle$, $\mathbf{v} = \langle 0, w, 0\rangle$, $\mathbf{w} = \langle 0, 0, h\rangle$
26. $\mathbf{u} = \langle l, 0, 0\rangle$, $\mathbf{v} = \left\langle \frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}, 0\right\rangle$, $\mathbf{w} = \left\langle 0, \frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}}\right\rangle$

27. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Use (2) to show that $\mathbf{v} \times \mathbf{u}$ is the same as $-(\mathbf{u} \times \mathbf{v})$.

28. Show that

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||^2 \, ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

29. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{a} = \langle a_1, a_2, a_3\rangle$. Show that

$$\mathbf{a} \times (\mathbf{u} + \mathbf{v}) = \mathbf{a} \times \mathbf{u} + \mathbf{a} \times \mathbf{v}$$

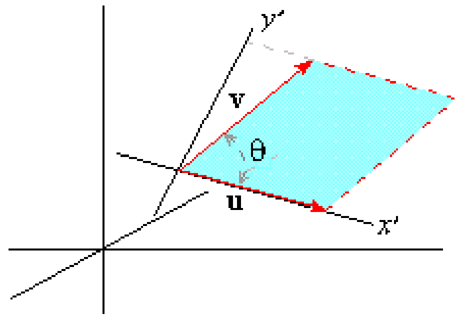
30. Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and let k be a number. Show that

$$k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} \qquad \textit{and} \qquad k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$$

31. Use the $x'y'$ coordinate system to show that the dot product of two vectors **u** and **v** is

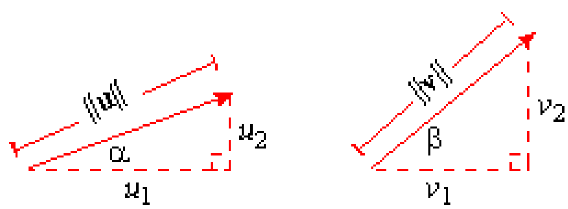
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \ \|\mathbf{v}\| \ \cos(\theta)$$

How would you use projections to define the y' axis in the $x'y'$ coordinate system?



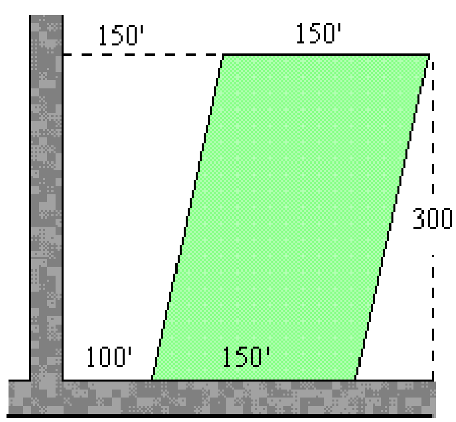
32. Let **u** and **v** be in the *polar forms* (in the $x'y'$ coordinate system in exercise 31) given by

$$\begin{aligned}\mathbf{u} &= \langle \|\mathbf{u}\| \cos(\alpha), \|\mathbf{u}\| \sin(\alpha), 0\rangle \\ \mathbf{v} &= \langle \|\mathbf{v}\| \cos(\beta), \|\mathbf{v}\| \sin(\beta), 0\rangle\end{aligned}$$



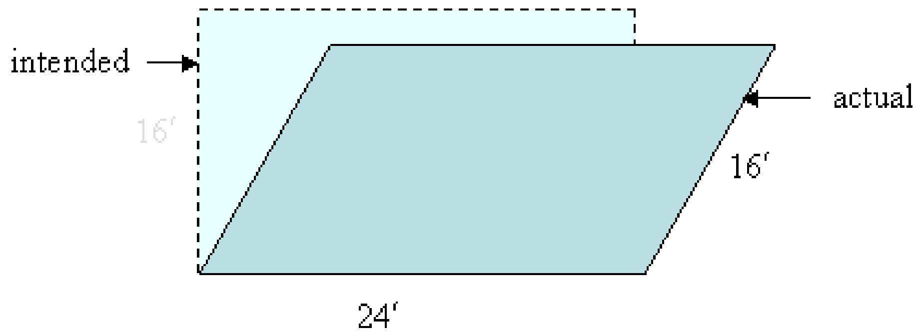
Show that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \ \|\mathbf{v}\| \ \sin(\theta)$, where $\theta = \beta - \alpha$.

33. Jane buys a corner lot which is 100 feet wide and 100 feet deep at the road, but is 150 wide and 130 feet deep at the lot's back corner.



What is the area of Jane's lot in acres? (Hint: 1 acre = $43,560$ feet^2).

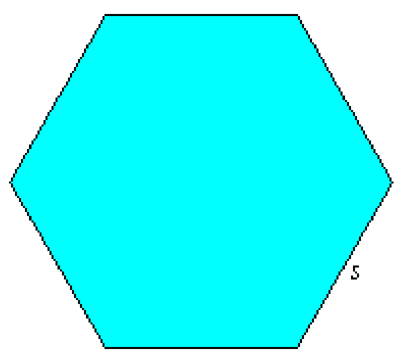
34. Roy intended to pour a $24'$ by $16'$ rectangular concrete patio, but he didn't check closely for square and instead ended up with a deck in the form shown below



Which has more area, the patio Roy intended to build or the one he ended up with?

35. Write to Learn: Write a short essay in which you develop the area formula for a triangle whose vertices have coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3), respectively. Write the essay as if you were instructing a group of surveyors.

36. Write to Learn: Suppose we are given that the length of a regular hexagon is s.



Write a short essay deriving a formula for the hexagon as a function of s and explaining how you arrived at that formula.

37. Sketch the parallelpiped spanned by $\mathbf{u} = \langle 1,0,0\rangle$, $\mathbf{v} = \langle 1,1,0\rangle$, and $\mathbf{w} = \langle 0,0,1\rangle$ assuming that all vectors have initial points at the origin.

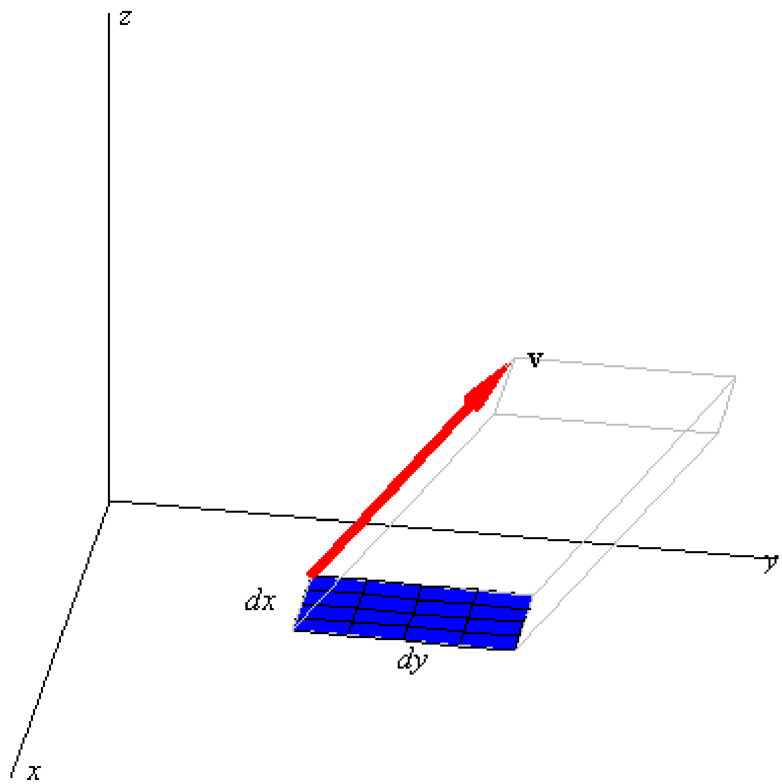
1. (a) Calculate the volume of the parallelpiped using $Volume = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.
   (b) Calculate the volume of the parallelpiped using $Volume = |(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$. Why should you expect the same result as in (a)?
   (c) How might you have obtained the result without the triple scalar product by "slicing and rearranging"?

38. Sketch the parallelpiped spanned by $\mathbf{u} = \langle 2,0,0 \rangle$, $\mathbf{v} = \langle 0,2,0 \rangle$, and $\mathbf{w} = \langle 1,1,1 \rangle$ assuming that all vectors have initial points at the origin.

1. (a) Calculate the volume of the parallelpiped using $Volume = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

   (b) Calculate the volume of the parallelpiped using $Volume = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$. Why should you expect the same result as in (a)?

39. Write to Learn: Suppose that over a short period of time, dt, a rectangle in the xy-plane with length dx and width dy moves from the initial point to the terminal point of a vector $\langle adt, bdt, cdt \rangle$, where a, b, and c are numbers.



Write a short essay deriving a formula for the volume of the parallelpiped swept out by the rectangle over that short time period.

40. What is the volume of the prism shown in the figure below?