Vectors

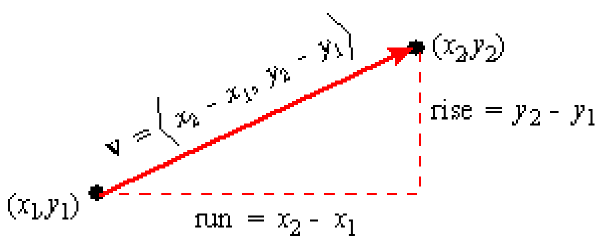
Part 1: Vectors in the Plane

Now that we have studied calculus in a 2 dimensional setting, our next step is to extend calculus to 3 or more dimensions. However, a slope allows only a rise and a run, and as a result, slope is inherently a two dimensional concept. Thus, we begin a concept that is the same across any number of dimensions, the concept of a *vector.*

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the xy-plane, we define the vector between them to be

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle \tag{1}$$

The points P_1 and P_2 are called, respectively, the *initial point* and the *terminal point* of the vector $\mathbf{v}$, and thus, $\mathbf{v}$ is often represented by an arrow beginning at a point P_1 and ending at P_2..



1.1.1: $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$

In general, ordinary numbers such as $1, \pi$, and 3.28 are called *scalars.* A *2-dimensional vector* is a grouping of two scalars a and b in the form $\mathbf{v} = \langle a, b \rangle$. The scalars a and b are known as the *components* of $\mathbf{v} = \langle a, b \rangle$.

EXAMPLE 1 Find the vector $\mathbf{u}$ with initial point $P_1(2, 4)$ and final point $P_2(5, 6)$.

Solution: To find $\mathbf{u}$, we apply (1) in the form

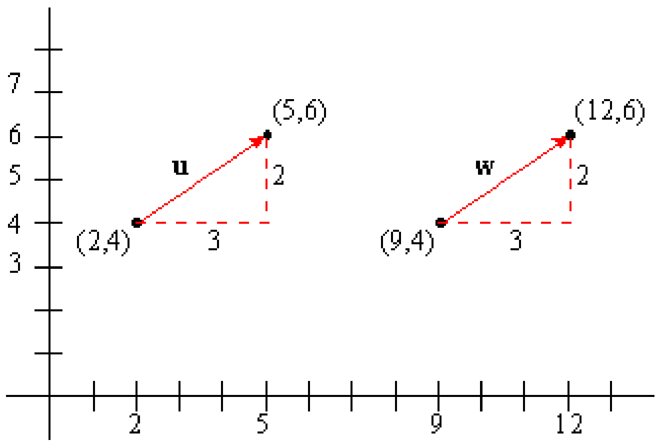
$$\mathbf{u} = \overrightarrow{P_1P_2} = \langle 5 - 2, 6 - 4 \rangle = \langle 3, 2 \rangle$$

EXAMPLE 2 Find the vector $\mathbf{w}$ with initial point $P_3(9, 4)$ and final point $P_4(12, 6)$.

Solution: To do so, we apply (1) to P_3 and P_4 to obtain

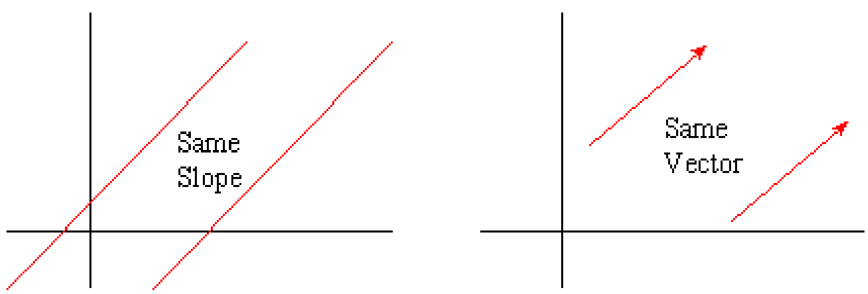
$$\mathbf{w} = \overrightarrow{P_3P_4} = \langle 12 - 9, 6 - 4 \rangle = \langle 3, 2 \rangle$$

Notice that the vectors **u** and **w** are the same, as is further illustrated in figure 1.1.2 :



1.1.2: $\mathbf{u} = \mathbf{w} = \langle 3, 2 \rangle$

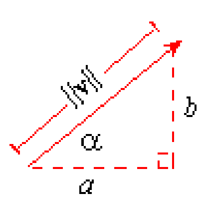
That is, two vectors are the same if they have the same components, which means that the translation of a vector results in the same vector, just as the translation of a line results in a line with the same slope as the original.



1.1.3: Equivalent vectors have the same slope.

Alternatively, a 2-dimensional vector $\mathbf{v} = \langle a, b \rangle$ may be defined to be a quantity with both a *magnitude* $\|\mathbf{v}\|$ and a *direction angle* α using trigonmetric relationships:

$$a = \|\mathbf{v}\| \cos(\alpha)$$
$$b = \|\mathbf{v}\| \sin(\alpha)$$



The Pythagorean theorem thus implies that

$$||\mathbf{v}|| = \sqrt{a^2 + b^2}$$

Moreover, the quantity $\cos(\alpha) = a/\,||\mathbf{v}||$ is the *direction cosine* of $\mathbf{v}$, and

$$\mathbf{v} = \langle \|\mathbf{v}\| \cos(\alpha), \|\mathbf{v}\| \sin(\alpha) \rangle$$

is the *polar form* of the vector $\mathbf{v}$.

EXAMPLE 3 What is the magnitude and direction angle of

$$\mathbf{v} = \langle 3, 2 \rangle$$

Solution: The magnitude is

$$||\mathbf{v}|| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

The direction angle follows from the direction cosine:

$$\cos(\alpha) = \frac{3}{\sqrt{13}}, \quad so\ \alpha = \cos^{-1}\left(\frac{3}{\sqrt{13}}\right) = 0.588\ radians$$

In degree measure, the direction angle is

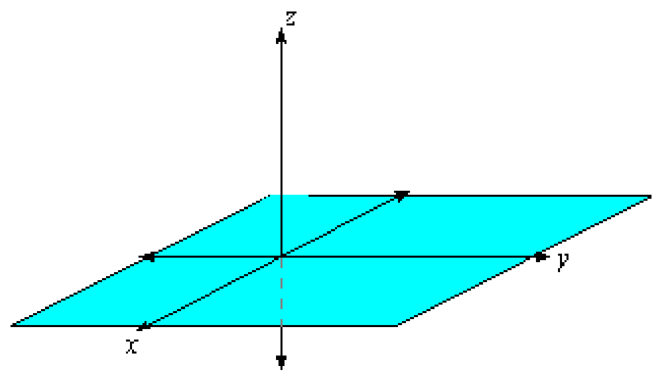
$$\alpha = 0.588\ radians \cdot \left(\frac{180°}{\pi\ radians}\right) = 33.6899°$$

It follows that vectors with the same direction angle are parallel, and that vectors with the same magnitude and direction are identical.

Check your reading: How does a slope differ from a vector?

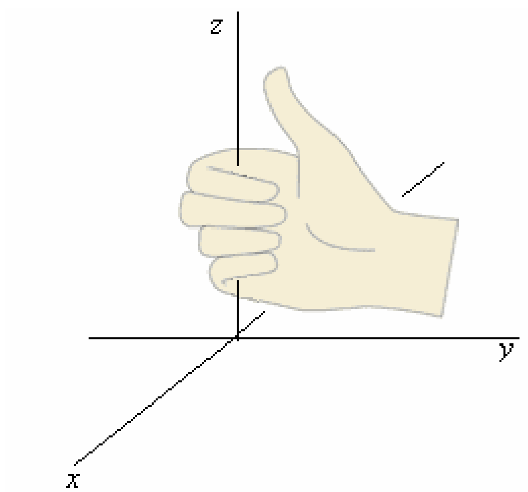
Part 2: Vectors in 3 dimensions

Three dimensional space, which is often denoted by $\mathbb{R}^3$, is often modeled as a horizontal xy-plane with a vertical z-axis intersecting the plane at their origins.



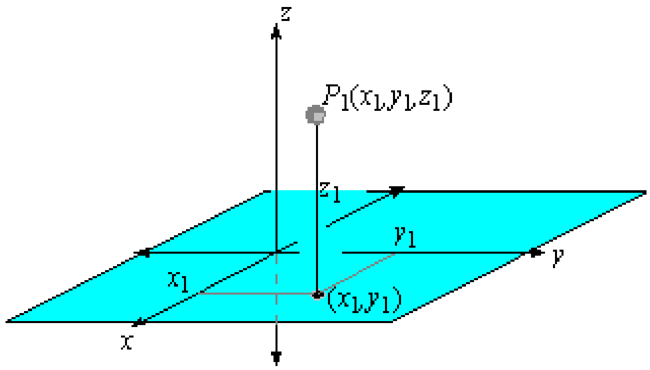
1.2.1: Three dimensional space

We assume that positive z-coordinates are above the xy-plane. Thus, the coordinate system is *right-handed* because if the fingers of the right hand are wrapped about the z-axis in the counter-clockwise direction (i.e., positive x is swept toward positive y), then the thumb of the right hand points in the positive z-direction.



1.2.2: Right handed orientation of xyz coordinate system.

The point $P_1\,(x_1, y_1, z_1)$ in $\mathbb{R}^3$ is the point that has a vertical displacement of z_1 above the point $(x_1, y_1, 0)$. Points in the xy-plane are of the form $(x_1, y_1, 0)$, which we often simply write as (x_1, y_1).



1.2.3: (x_1, y_1, z_1) is displaced z_1 units vertically from the point $(x_1, y_1, 0)$.

If a point is in the xy-plane, then we will continue to write it with only two coordinates.

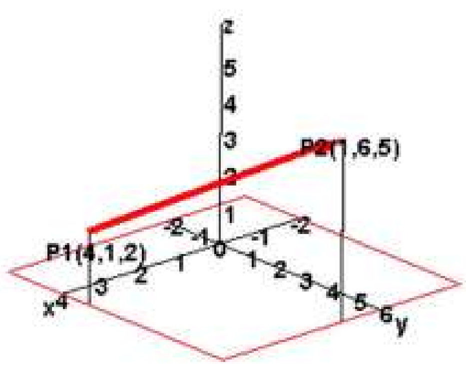
Given points $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$ in $\mathbb{R}^3$, we define the vector between them to be

$$\mathbf{v} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{2}$$

Notice that in 3 dimensions a vector is of the form $\mathbf{v} = \langle a, b, c \rangle$ where a is a run, b is a shift left or right, and c is a rise. Thus, vectors generalize to 3 dimensions whereas slopes do not.

To aid in visualization, this text will include many figures which are interactive. These figures are prepared using the packages Javaview and LiveGraphics3D. For example, if you "click and drag" on the figure in example 3, it will rotate. Right-clicking produces more options for interactivity.

EXAMPLE 4 Find the vector with initial point $P_1\,(4, 1, 2)$ and terminal point $P_2\,(1, 6, 5)$ .d



Solution: To do so, we use (2) to obtain

$$\mathbf{v} = \langle 1-4, 6-1, 5-2 \rangle = \langle -3, 5, 3 \rangle$$

Given 3-dimensional vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, we define

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

and for a scalar k we define $k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$. Two dimensional vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ are actually 3-dimensional vectors of the form $\mathbf{v} = \langle v_1, v_2, 0 \rangle$, so that the same arithmetic applies to both 2-dimensional vectors and 3-dimensional vectors.

The *basic vectors* are defined either by

$$\mathbf{i} = \langle 1, 0, 0 \rangle\,, \quad \mathbf{j} = \langle 0, 1, 0 \rangle\,, \quad \textit{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

We say that the basic vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a *basis* for $\mathbb{R}^3$ in that any vector $\langle a, b, c \rangle$ can be written as

$$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

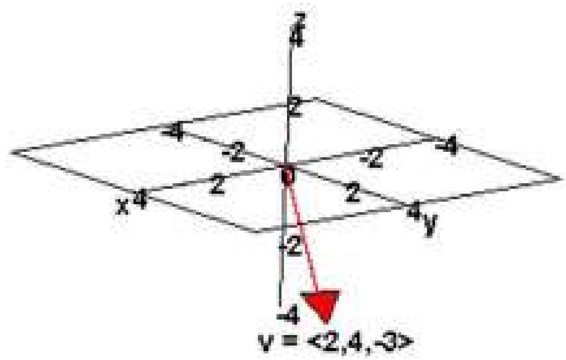
Conversely, we say that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a *linear combination* of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

EXAMPLE 5 Convert the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ to component form, and then sketch the vector.

Solution: The definitions of the basis vectors lead to

$$\begin{aligned} 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} &= 2\langle 1, 0, 0 \rangle + 3\langle 0, 1, 0 \rangle - \langle 0, 0, 1 \rangle \\ &= \langle 2, 0, 0 \rangle + \langle 0, 3, 0 \rangle - \langle 0, 0, 1 \rangle \\ &= \langle 2+0+0, 0+3+0, 0+0-1 \rangle \\ &= \langle 2, 3, -1 \rangle \end{aligned}$$

Thus, $\mathbf{v} = \langle 2, 3, -1 \rangle$, which is shown with initial point at the origin in the figure below.



Note: The use of **i**, **j**, and **k** is very common, but other notations are also used. For example, in some settings the basic unit vectors may be denoted

$$\mathbf{e}_x = \langle 1, 0, 0 \rangle, \quad \mathbf{e}_y = \langle 0, 1, 0 \rangle, \quad \textit{and} \quad \mathbf{e}_z = \langle 0, 0, 1 \rangle$$

or capital letters **I**, **J**, and **K** may be used in place of lower case.

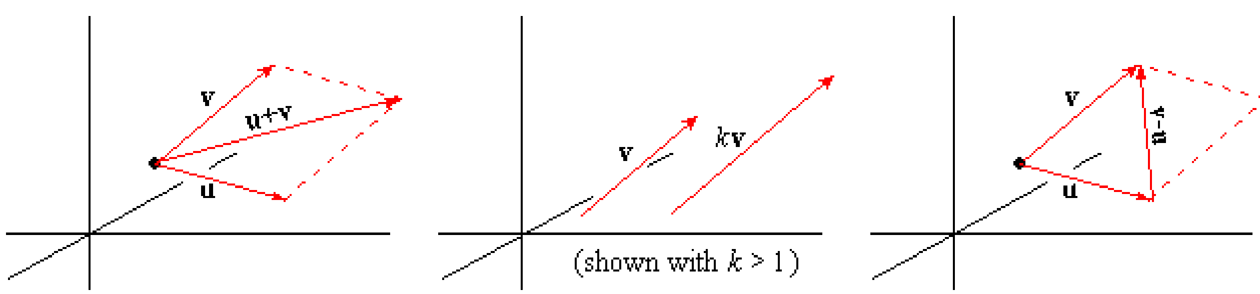
Check your Reading: What is the component form for $\mathbf{v} = 2\mathbf{i} + 3\mathbf{J} - \mathbf{e}_z$?

Part 3: Vector Arithmetic

Addition and scalar multiplication implies that the difference of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

The *parallelogram law* (which we prove in the exercises) says that the sum $\mathbf{u} + \mathbf{v}$ is the *main diagonal* of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, while $\mathbf{u} - \mathbf{v}$ is the *off-diagonal* of the parallelogram.



1.3.1: Arithmetic of Vectors

If $\mathbf{u} = k\mathbf{v}$ for some nonzero scalar k, then $\mathbf{u}$ and $\mathbf{v}$ are said to be *scalar multiples* of each other. Geometrically, $\mathbf{u}$ and $\mathbf{v}$ are parallel only if there is a number k such that $\mathbf{u} = k\mathbf{v}$ with $k \neq 0$.

EXAMPLE 6 Find $\mathbf{u}+\mathbf{v}$ and $2\mathbf{v}$ when $\mathbf{u} = \langle 3, 4, -2\rangle$ and $\mathbf{v} = \langle 0, -4, 0\rangle$.

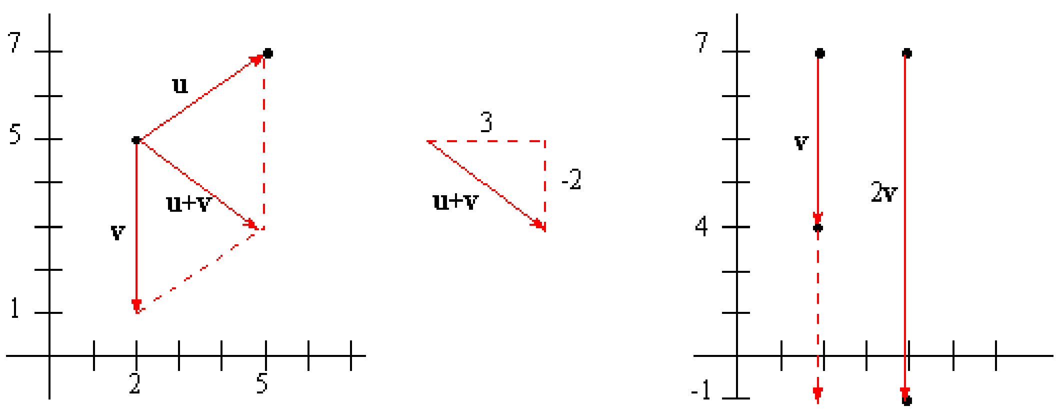
Solution: Their sum is given by

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \langle 3, 4, -2\rangle + \langle 0, -4, 0\rangle \\ &= \langle 3+0, 4-4, -2-0\rangle \\ &= \langle 3, 0, -2\rangle \end{aligned}$$

Moreover, multiplication of $\mathbf{v} = \langle 0, -4\rangle$ by the scalar 2 yields

$$2\mathbf{v} = 2\langle 0, -4, 0\rangle = \langle 0, -8, 0\rangle$$

The sum and scalar multiplication are shown geometrically below:



1.3.2: Arithmetic Examples

The $\mathbf{0}$ vector is defined to be $\mathbf{0} = \langle 0, 0, 0\rangle$. It is shown in the exercises that the arithmetic of vectors has the following properties:

Theorem 1.1: If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and if k, m are scalars, then

(1)	$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$	(5)	$k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$
(2)	$\mathbf{u}+\mathbf{0} = \mathbf{u}$	(6)	$(k+m)\mathbf{u} = k\mathbf{u}+m\mathbf{u}$
(3)	$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$	(7)	$(km)\mathbf{u} = k(m\mathbf{u})$
(4)	$\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$		

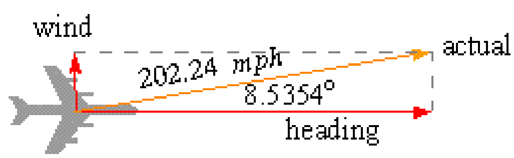
The properties in Theorem 1.1 along with the concepts of magnitude and direction allow vectors to be used in many different applications.

EXAMPLE 7 An airplane **heads** due east at 200 mph through a crosswind blowing due north at 30 mph, and the superposition (i.e., sum) of these two velocities is the airplane's actual velocity vector. What is the airplane's actual speed and direction?

Solution: The heading velocity vector is $\langle 200, 0\rangle$ and the wind velocity vector is $\langle 0, 30\rangle$. Thus, the actual veclocity of the airplane is

$$\mathbf{v} = \langle 200, 0\rangle + \langle 0, 30\rangle = \langle 200, 30\rangle$$

which is the main diagonal of the parallelogram (in this case, a rectangle) formed by the heading and wind vectors.



1.3.3: Airplane's actual heading is approximately 8.5° from its apparent heading.

The airplane's actual speed is the magnitude of $\mathbf{v}$:

$$||\mathbf{v}|| = \sqrt{200^2 + 30^2} = 202.24 \quad mph$$

The actual direction can be found from the fact that $||\mathbf{v}|| \cos(\alpha) = a$, which in our case yields

$$202.24 \cos(\alpha) = 200, \quad \alpha = \cos^{-1}\left(\frac{200}{202.24}\right) = 0.14897 \; radians$$

This is about $\alpha = 0.14897\,(180/\pi) = 8.535\,4^\circ$ north of due east.

In general, a vector is an element of a *Vector Space*, where a *Vector Space* is a set with addition and scalar multiplication operations satisfying (1) - (7) in theorem 1.1. For example, $\mathbb{R}^3$ is the vector space of all possible 3-dimensional vectors, and a given vector $\mathbf{v} = \langle a, b, c\rangle$ is an *element* in the vector space $\mathbb{R}^3$.

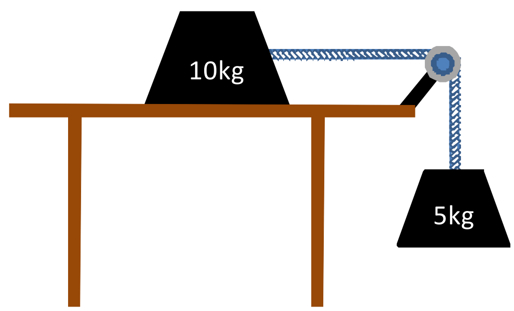
Check your Reading: *Is* $2\mathbf{v}$ *the same as* $\mathbf{v} + \mathbf{v}$ *? Explain.*

Part 4: Free Body Diagrams

A force $\mathbf{F}$ acting on a body (i.e., object) is represented mathematically by a vector, in that it has a magnitude $\|\mathbf{F}\|$ which is applied in a given direction. The magnitude $\|\mathbf{F}\|$ is measured in *Newtons,* where 1 Newton is the force required to accelerate a 1 kg mass at a rate of 1 meter per second per second (i.e, $1\ N = 1\ kg \cdot m/\sec^2$).

A *force diagram,* which is also known as a *free body diagram,* is a sketch in which all the force vectors acting on an object are drawn with their initial points at the location of the object. The *net force* acting on the body is the sum of all the forces in the force diagram.

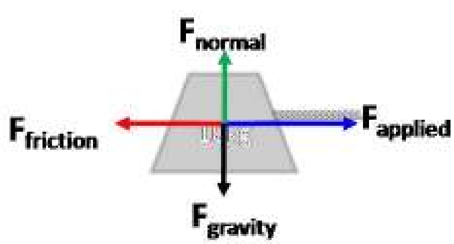
blue*EXAMPLE 8* blackA 10kg block resting on a table is attached by a "massless" rope to a 5 kg block hanging off the end (with the rope running across a "frictionless" pulley).



1.4.1: 10 kg block is not moving.

Draw the free-body diagram and determine the net force if the 10kg block is experiencing a force of friction with a magnitude of 30 newtons.

Solution: To begin with, the force of gravity $\mathbf{F}_{gravity}$ is pulling the 10 kg block toward the earth, which via Newton's third law is countered by a force normal (i.e., perpendicular) to the table that resists the 10 kg blocks fall. If the block is not moving vertically, then $\mathbf{F}_{normal}$ and $\mathbf{F}_{gravity}$ must have the same magnitude so that they cancel each other out (i.e, no net force up or down).



1.4.2: Free body diagram for Figure 1.4.1.

Using Newton's law $F = ma$, we can show that the 5kg block is subject to a gravitational force of

$$\mathbf{F}_{5kg} = 5kg \cdot \left\langle 0, -9.8\frac{m}{\sec^2} \right\rangle = \left\langle 0, -49\frac{kgm}{\sec^2} \right\rangle$$

Since $\|\mathbf{F}_{5kg}\| = 49N$, the 5kg block applies a force of

$$\mathbf{F}_{applied} = 49\mathbf{i} \quad Newtons$$

to the 10kg block. Finally, the force of friction $\mathbf{F}_{friction}$ resists the horizontal motion of the block, so it is applied in the direction $-\mathbf{i}$, or

$$\mathbf{F}_{friction} = -30\mathbf{i} \; Newtons$$

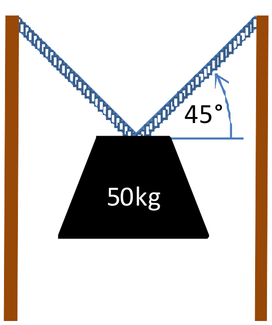
Thus, the net force is

$$\begin{aligned} \mathbf{F}_{net} &= \mathbf{F}_{gravity} + \mathbf{F}_{normal} + \mathbf{F}_{applied} + \mathbf{F}_{friction} \\ &= 0 + 49\mathbf{i} - 30\mathbf{i} \\ &= 19\mathbf{i} \end{aligned}$$

In example 8, a net force of $\mathbf{F}_{net} = 19\mathbf{i}$ means that the 10kg block is accelerating to the right at a rate of 1.9 meters per second per second. In contrast, had the friction had a magnitude of 49 Newtons, it would not be moving at all – i.e., it would be at rest with the 5kg block likewise at rest as it hangs suspended in midair.

If the net force acting on a body is 0, then the body is said to be at *equilibrium.* When it occurs, equilibrium often can be used to determine unknown quantitites in a free body diagram.

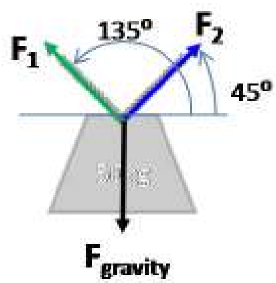
EXAMPLE 9 A 50kg block is suspended by 2 identical cables of the same length forming 45° angles with the horizontal.



1.4.3: Suspended block

If the block remains motionless over time, then what is the magnitude of the tensile force exerted by each of the cables on the block?

Solution: In this case, the free body diagram of the 50 kg mass is given by figure 1.4.4.



1.4.4: Free Body diagram for 1.4.3.

The force of gravity is

$$\mathbf{F}_{gravity} = 50kg \cdot \left\langle 0, -9.8\frac{m}{\sec^2} \right\rangle = \langle 0, -490N \rangle$$

If we let F denote the (unknown) magnitude of the force exerted by a cable, then

$$\begin{aligned} \mathbf{F}_2 &= \langle F\cos(45^\circ), F\sin(45^\circ) \rangle = \left\langle \frac{F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle \\ \mathbf{F}_1 &= \langle F\cos(135^\circ), F\sin(135^\circ) \rangle = \left\langle \frac{-F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle \end{aligned}$$

Consequently, the net force is

$$\begin{aligned} \mathbf{F}_{net} &= \mathbf{F}_{gravity} + \mathbf{F}_2 + \mathbf{F}_1 \\ &= \langle 0, -490 \rangle + \left\langle \frac{F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle + \left\langle \frac{-F}{\sqrt{2}}, \frac{F}{\sqrt{2}} \right\rangle \\ &= \left\langle 0, -490 + \frac{2F}{\sqrt{2}} \right\rangle \end{aligned}$$

Since $\mathbf{F}_{net} = \mathbf{0}$ (equilibrium), we have

$$\frac{2F}{\sqrt{2}} = 490, \quad F = \frac{490\sqrt{2}}{2} = 346.48\ N$$

Exercises

Find the vector with initial point P_1 and final point P_2. Sketch the result in either the xy-plane or in R^3, whichever is appropriate. Also, write the vector in basis format (i.e., in terms of ***i****,* ***j****, and* ***k****.)*

1.	$P_1(2,3), P_2(3,5)$	2.	$P_1(3,7), P_2(-4,-11)$
3.	$P_1(7,9,2), P_2(3,7,0)$	4.	$P_1(7,9,2), P_2(3,7,0)$
5.	$P_1(-4,-17,1), P_2(-3,3,5)$	6.	$P_1(0,0,0), P_2(1,3,1)$
7.	$P_1(8,-10,3), P_2(1,10,7)$	8.	$P_1(-1,-3,-1), P_2(0,0,0)$
9.	$P_1(a,b,c), P_2(a+1,b+2,c+3)$	10.	$P_1(a,2,c), P_2(a+1,5,c+3)$

Find $\mathbf{u}+\mathbf{v}$ *and* $\mathbf{u}-\mathbf{v}$ *for the given vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, and then sketch* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{u}+\mathbf{v}$ *and* $\mathbf{u}-\mathbf{v}$.

11.	$\mathbf{u}=\langle 1,3\rangle, \mathbf{v}=\langle 2,5\rangle$	12.	$\mathbf{u}=\langle 1,3\rangle, \mathbf{v}=\langle 2,5\rangle$
13.	$\mathbf{u}=\langle 1,-2\rangle, \mathbf{v}=\langle -3,5\rangle$	14.	$\mathbf{u}=\langle 0,1\rangle, \mathbf{v}=\langle 1,0\rangle$
15.	$\mathbf{u}=\langle -1,-1\rangle, \mathbf{v}=\langle 1,1\rangle$	16.	$\mathbf{u}=\langle 2,3\rangle, \mathbf{v}=\langle 4,6\rangle$
17.	$\mathbf{u}=\langle 1,0,0\rangle, \mathbf{v}=\langle 0,1,0\rangle$	18.	$\mathbf{u}=\langle 0,1,0\rangle, \mathbf{v}=\langle 0,0,1\rangle$
19.	$\mathbf{u}=\langle 2,1,0\rangle, \mathbf{v}=\langle 0,3,5\rangle$	20.	$\mathbf{u}=\langle 1,-2,3\rangle, \mathbf{v}=\langle 2,5,-4\rangle$

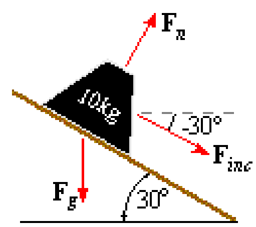
The following vectors represent forces acting on a single object in space. Draw the Force Diagram and determine the net force acting on the object. Is the object at equilibrium?

21. $\mathbf{F}_1=\langle 0,-9.8\rangle, \quad \mathbf{F}_2=\langle -3,3\rangle, \mathbf{F}_3=\langle 3,3\rangle$
22. $\mathbf{F}_1=\langle 0,-98\rangle, \quad \mathbf{F}_2=\langle -30,30\rangle, \mathbf{F}_3=\langle 30,30\rangle$
23. $\mathbf{F}_1=\langle 0,-9.8\rangle, \quad \mathbf{F}_2=\langle 0,9.8\rangle$
24. $\mathbf{F}_1=\langle 0,-15\rangle, \quad \mathbf{F}_2=\langle 0,15\rangle$
25. $\mathbf{F}_1=\langle 0,0,-9.8\rangle, \quad \mathbf{F}_2=\langle 10,0,10\rangle, \mathbf{F}_3=\langle 0,10,10\rangle$
26. $\mathbf{F}_1=\langle 1,0,-1\rangle, \quad \mathbf{F}_2=\langle -1,1,0\rangle, \mathbf{F}_3=\langle 0,-1,1\rangle$

27. An airplane heads due north at 300 mph through a cross-wind blowing due east at 50 mph. What is the actual speed and direction of the airplane?
28. An airplane heads due east at 300 mph through a tailwind whose velocity is given by

$$\mathbf{w}=\langle 20,20\rangle$$

How fast is the tailwind blowing? In what direction? How fast is the plane flying? In what direction?
29. Gravity acting on a 10 kg mass produces a force of $\mathbf{F}_g=\langle 0,-98\rangle$ newtons. If the mass is suspended from 2 wires which both form 30° angles with the horizontal, then what forces of tension are required in order for the mass to hang motionless over time?
30. Repeat exercise 29 given angles of 45° instead of 30°.
31. A 10 kg block sits on a board inclined to an angle of 30° with the horizontal.



What are $\mathbf{F}_{inc}$ (the force along the incline) and $\mathbf{F}_n$ (the force normal)?
32. Repeat exercise 31 given an incline of $10°$ instead of $30°$.
33. Given $P_1(1,2,1)$ and $\mathbf{v} = \langle 3,2,1 \rangle$, find the point P_2 for which

$$\overrightarrow{P_1P_2} = \mathbf{v}$$

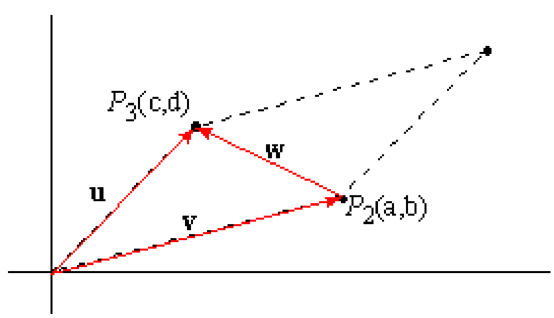
34. Given $P_2(5,5,5)$ and $\mathbf{v} = \langle 5,-2,1 \rangle$, find the point P_1 for which

$$\overrightarrow{P_1P_2} = \mathbf{v}$$

35. In this exercise, we explore the difference of 2 vectors.

1. (a) Let $\mathbf{v}$ be the vector with initial point $P_1(3,5)$ and final point $P_2(5,1)$. What is $\mathbf{v}$?
   (b) Let $\mathbf{u}$ be the vector with initial point $P_1(3,5)$ and final point $P_3(6,7)$. What is $\mathbf{u}$?
   (c) Compute $\mathbf{u} - \mathbf{v}$ and show that it is the same as the vector from $P_2(3,5)$ to $P_3(6,7)$.
   (d) Sketch a graph of the system.

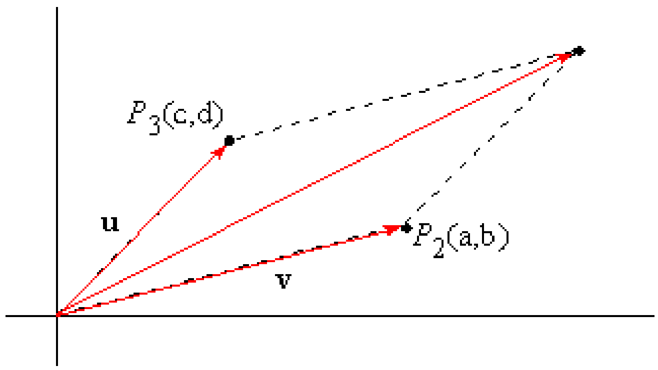
36. Let's prove that in general, the difference of 2 two dimensional vectors is the cross-diagonal of the parallelogram they form. Because vectors are translation invariant, we need only do so for vectors with initial points at the origin.



Let $\mathbf{v}$ be the vector from $P_1(0,0)$ to $P_2(a,b)$, and let $\mathbf{u}$ be the vector from $P_1(0,0)$ to $P_3(c,d)$. What is the vector from $P_2(a,b)$ to $P_3(c,d)$? How is it related to $\mathbf{u} - \mathbf{v}$?
37. In this exercise, we explore the sum of 2 vectors.

1. (a) Let **v** be the vector with initial point $P_1(0,0)$ and final point $P_2(5,1)$. What is **v**?
   (b) Let **u** be the vector with initial point $P_1(0,0)$ and final point $P_3(3,5)$. What is **u**?
   (c) Compute $\mathbf{u}+\mathbf{v}$, and then sketch a graph of the system, including the parallelogram formed by **u** and **v**.

38. Let's prove that in general, the sum of 2 two dimensional vectors is the main diagonal of the parallelogram they form. Because vectors are translation invariant, we need only do so for vectors with initial points at the origin.

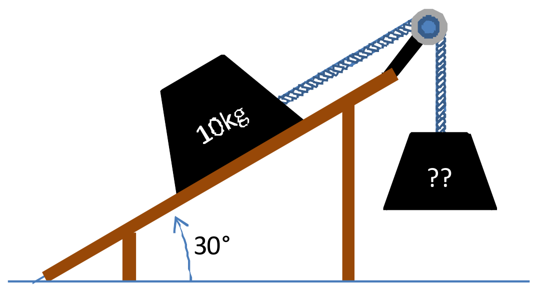


1. (a) Explain why the line through $(0,0)$ and (a,b) is parallel to the line through (c,d) and $(a+c,b+d)$.
   (b) Explain why the line through $(0,0)$ and (c,d) is parallel to the line through (a,b) and $(a+c,b+d)$. Why does this imply that $(a+c,b+d)$ is the fourth vertex of the parallelogram with vertices $(0,0)$, (a,b), and (c,d).
   (c) Let **v** be the vector from $P_1(0,0)$ to $P_2(a,b)$, and let **u** be the vector from $P_1(0,0)$ to $P_3(c,d)$. Find $\mathbf{u}+\mathbf{v}$ and relate it to the diagram above?

39. Let $\mathbf{u}=\langle u_1,u_2,u_3\rangle$, $\mathbf{v}=\langle v_1,v_2,v_3\rangle$, and $\mathbf{w}=\langle w_1,w_2,w_3\rangle$. Prove properties 1, 3, and 4 of theorem 1.1

40. Let k,m be scalars and let $\mathbf{u}=\langle u_1,u_2,u_3\rangle$, $\mathbf{v}=\langle v_1,v_2,v_3\rangle$. Prove properties 5, 6, and 7 of theorem 1.1

41. A 10 kg block *is at rest* on a *frictionless* incline at 30° to the horizontal, held in place by a rope attached to a mass hanging off the end of the incline.

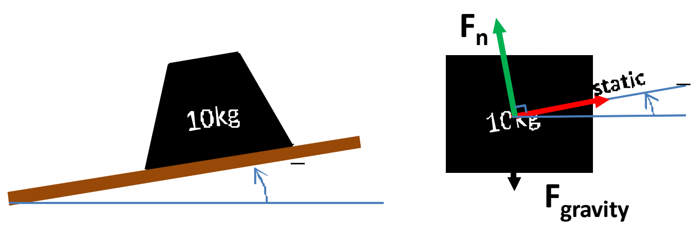


Exercise 41: Block on frictionless incline.

What is the mass of the block at the other end of the rope?
42. Class Discussion: A block at rest on an incline experiences a force of static friction $\mathbf{F}_{static}$ with a maximum magnitude of

$$\|\mathbf{F}_{static}\| = \mu_s \|\mathbf{F}_n\| \tag{3}$$

where $\mathbf{F}_n$ is the force normal (perpendicular) to the incline pushing "up" against the block and μ_s is the coefficient of static friction of the incline's surface.



Use the diagram to explain how you could estimate μ_s experimentally by slowly lifting the end of the incline (i.e. varying α) until the block begins to slide
43. A *vector space* is a set V which is closed under an addition operation and a scalar multiplication operation which jointly satisfy theorem 1.1. Show that the set of functions

$$V = \{f \ : \ f \text{ is continuous on } [0,1]\ \}$$

is a vector space by showing that it is closed under addition, closed under multiplication by a number, and satisfies the properties in theorem 1.1.
44. A *vector space* is a set V which is closed under an addition operation and a scalar multiplication operation which jointly satisfy theorem 1.1. Show that the set of functions

$$V = \left\{ f(x) = a + bx + cx^2, \ \textit{where } a, b, c \textit{ are numbers} \right\}$$

is a vector space by showing that it is closed under addition, closed under multiplication by a number, and satisfies the properties in theorem 1.1.