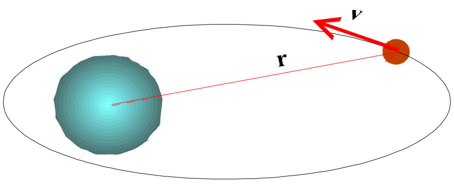
Kepler's Problem

Suppose a satellite with mass m in orbit about the earth has a position $\mathbf{r}(t)$ and a velocity $\mathbf{v}(t)$ at time t.

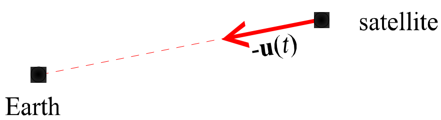


If $r = ||\mathbf{r}||$ denotes the distance to the satellite at time t and if we assume that the earth is a perfect sphere centered at the origin with a radius of $R_e = 3963.21$ miles, then the magnitude of the earth's gravitational force $\mathbf{F}(\mathbf{r})$ acting on the object satisfies an *inverse square law* of the form

$$\|\mathbf{F}(\mathbf{r})\| = \frac{mk}{r^2}$$

where $k = 95194.14\ \frac{mi^3}{\sec^2}$ for the earth's gravitational field.

The direction of $\mathbf{F}$ is in the opposite direction of $\mathbf{r}$, which is in the direction of $\mathbf{u}(t) = \mathbf{r}(t)\,/$



That is, if we define the unit vector

$$\mathbf{u}(t) = \frac{\mathbf{r}(t)}{||\mathbf{r}(t)||} = \frac{1}{r}\mathbf{r}(t)$$

then the direction of the gravitational force is $-\mathbf{u}(t)$. Thus, the force of gravity is

$$\mathbf{Force} = magnitude \cdot direction = \frac{mk}{r^2} \cdot \frac{-\mathbf{r}}{r} = -\frac{mk}{r^3}\mathbf{r}$$

Moreover, since **Force** $=$*mass* $\times$ *acceleration*, we have

$$m\mathbf{r}'' = -\frac{mk}{r^3}\mathbf{r}$$

which reduces to

$$\mathbf{r}'' = \frac{-k}{r^3}\mathbf{r} \tag{0.1}$$

where for motion about the earth, $k = 95,194.14\ \frac{mi^3}{\sec^2}$. The equation (0.1) is called *Kepler's problem,* and solutions to (0.1) are called *Keplerian orbits.* As we saw in the last section, *uniform circular motion* is a special type of Keplerian orbit.

EXAMPLE 1 A satellite is 100 miles above the earth and has a period of $T = 5234.14$ seconds, or $T = 1$ hour, 27 minutes and 14 seconds. Show that its parametrization is a Keplerian orbit.

Solution: Since the satellite's orbit is given by

$$\mathbf{r}(t) = 4063.21 \left\langle \cos\left(\frac{2\pi}{5234.14} t\right), \sin\left(\frac{2\pi}{5234.14} t\right)\right\rangle$$

its velocity and acceleration are

$$\begin{aligned}
\mathbf{r}'(t) &= 4063.21 \left\langle -\sin\left(\frac{2\pi}{5234.14} t\right), \cos\left(\frac{2\pi}{5234.14} t\right)\right\rangle \frac{2\pi}{5234.14} \\
\mathbf{r}''(t) &= 4063.21 \left\langle -\cos\left(\frac{2\pi}{5234.14} t\right), -\sin\left(\frac{2\pi}{5234.14} t\right)\right\rangle \left(\frac{2\pi}{5234.14}\right)^2
\end{aligned}$$

However, the acceleration can be rewritten as

$$\mathbf{r}''(t) = -\mathbf{r}(t) \cdot \left(\frac{2\pi}{5234.14}\right)^2 = -\left(1.441017 \times 10^{-6}\right) \mathbf{r}(t)$$

Consider now that

$$\frac{-k}{r^3}\mathbf{r}(t) = \frac{95194.14}{(4063.21)^3}\mathbf{r}(t) = -\left(1.419065 \times 10^{-6}\right) \mathbf{r}(t)$$

which is very close to the value of $\mathbf{r}''(t)$.

Check your Reading: Why does the r^3 term appear in (0.1) if it is called the inverse *square* law?

Conservation of Angular Momentum

Let's derive some of the properties of the solutions to Kepler's problem. To begin with, let us notice that

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt}$$

Since $d\mathbf{r}/dt = \mathbf{v}$ is velocity and since $d\mathbf{v}/dt = \mathbf{r}''$ is acceleration, (0.1) implies that

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{r}'' \\
&= 0 + \mathbf{r} \times \left(\frac{-k}{r^3}\mathbf{r}\right) \\
&= \frac{-k}{r^3}(\mathbf{r} \times \mathbf{r}) \\
&= 0
\end{aligned}$$

That is, the time derivative of $\mathbf{r} \times \mathbf{v}$ vanishes, so that $\mathbf{r} \times \mathbf{v}$ itself must be a constant vector. We let $\mathbf{L} = \langle L_1, L_2, L_3 \rangle$ denote this constant, and we say that $\mathbf{L}$ is the *angular momentum vector.* That is,

$$\mathbf{L} = \mathbf{r} \times \mathbf{v}$$

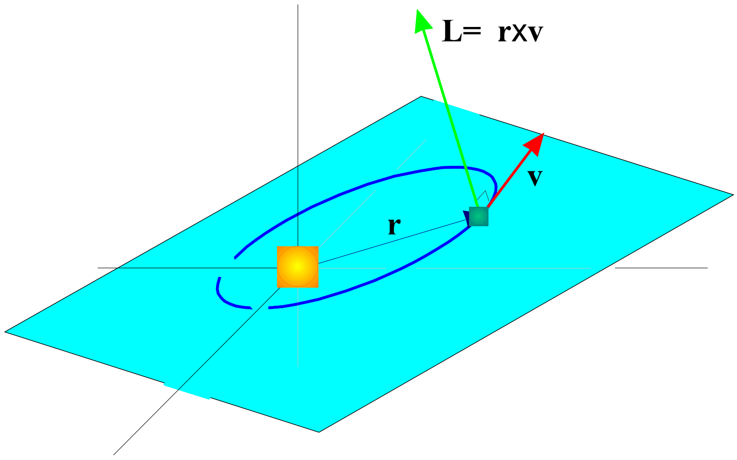
where the angular momentum vector $\mathbf{L}$ is constant. This result is more widely known as the *conservation of angular momentum.*

EXAMPLE 2 A satellite has initial position $\mathbf{r}_0 = \langle 3000, 4000, 0 \rangle$ and an initial velocity of $\mathbf{v}_0 = \langle 2, 2, 1 \rangle$. What is its angular momentum?

Solution: Since $\mathbf{L} = \mathbf{r}(t) \times \mathbf{v}(t)$ for all time t, we must also have $\mathbf{L} = \mathbf{r}(0) \times \mathbf{v}(0)$. Thus,

$$\mathbf{L} = \langle 3000, 4000, 0 \rangle \times \langle 2, 2, 1 \rangle = \langle 4000, -3000, -2000 \rangle$$

Since $\mathbf{r}$ and $\mathbf{v}$ must be perpendicular to $\mathbf{L}$ at all times, the fact that $\mathbf{L}$ is constant implies that $\mathbf{r}$ and $\mathbf{v}$ are in the plane with normal $\mathbf{L}$. That is, conservation of angular momentum implies that the motion of the satellite is in the plane with normal $\mathbf{L}$.



We can use this result to determine the *plane of motion* of a Keplerian orbit.

EXAMPLE 3 What is the equation of the plane of the orbit of the satellite in example 2?

Solution: The plane of motion must contain the point $\mathbf{r}(0) = \langle 3000, 4000, 0 \rangle$. Since $\mathbf{L} = \langle 4000, -3000, -2000 \rangle$ is normal to the plane, we have

$$4000(x - 3000) - 3000(y - 4000) - 2000(z - 0) = 0$$

which reduces to $z = 2x - 1.5y$. Thus, the satellite's orbit must be in the plane $z = 2x - 1.5y$ for all time.

Check your Reading: What is the initial speed of the satellite in the previous example?

Conservation of Energy

Let's find another conserved quantity–i.e., a quantity which is constant along a Keplerian orbit. Notice that

$$\frac{d}{dt}\frac{1}{2}v^2 = \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}\left(\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right) = \frac{1}{2} \cdot 2\mathbf{v} \cdot \mathbf{a}$$

since $d\mathbf{v}/dt$ is the acceleration $\mathbf{a}$ of the object. Likewise, let us notice that

$$\begin{aligned} \frac{d}{dt}\frac{k}{r} &= \frac{d}{dt}\frac{k}{(\mathbf{r}\cdot\mathbf{r})^{1/2}} \\ &= \frac{-1}{2}\frac{k}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) \end{aligned}$$

since $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$, and notice that this simplifies to

$$\begin{aligned} \frac{d}{dt}\frac{k}{r} &= \frac{-1}{2}\frac{k}{(\mathbf{r}\cdot\mathbf{r})^{3/2}}\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}+\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) \\ &= \frac{-1}{2}\frac{k}{r^3}(2\mathbf{v}\cdot\mathbf{r}) \\ &= \frac{-k}{r^3}\mathbf{v}\cdot\mathbf{r} \end{aligned}$$

since $d\mathbf{r}/dt$ is the velocity $\mathbf{v}$ of the object. Since $\mathbf{a} = \mathbf{r}''$, this means that

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{k}{r}\right) &= \frac{d}{dt}\frac{1}{2}v^2 - \frac{d}{dt}\frac{k}{r} \\ &= \mathbf{v}\cdot\mathbf{a} - \frac{-k}{r^3}\mathbf{v}\cdot\mathbf{r} \\ &= \mathbf{v}\cdot\left(\mathbf{r}'' - \frac{-k}{r^3}\mathbf{r}\right) \end{aligned}$$

Kepler's problem implies that $\mathbf{r}'' = -k\mathbf{r}/r^3$, so that $\mathbf{r}'' - -k\mathbf{r}/r^3 = 0$ and thus

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{k}{r}\right) = \mathbf{v}\cdot 0 = 0$$

Thus, the quantity $\frac{1}{2}v^2 - \frac{k}{r}$ is constant. That is, there is a constant H such that

$$H = \frac{1}{2}v^2 - \frac{k}{r} \tag{0.2}$$

Moreover, since $\frac{-k}{r}$ is the *potential energy* of the satellite, and since $\frac{1}{2}v^2$ is the *kinetic energy* of the satellite, the quantity H is the *total energy* or *Hamiltonian energy* of the satellite, and the equation (0.2) is called the *Law of Conservation of Energy.* Moreover, if $H < 0$, then that means that the potential energy is dominant and thus that the satellite cannot escape the earth's gravitational field. However, if $H > 0$, then the kinetic energy is dominant, and thus, the satellite will eventually escape from earth's gravitational field.

EXAMPLE 4 Find the total energy for the satellite which at time $t = 0$ is at the point $\mathbf{r}(0) = \langle 3000, 4000, 0\rangle$ with an velocity of $\mathbf{v}(0) = \langle 2, 2, 1\rangle$.

Solution: To begin with, we notice that

$$\begin{aligned} r(0) &= \|\mathbf{r}(0)\| = \sqrt{(3,000)^2 + (4,000)^2 + 0^2} = 5,000\ miles \\ v(0) &= \|\mathbf{v}(0)\| = \sqrt{2^2+2^2+1^2} = 3 \end{aligned}$$

Since $k = 95194.14\ \frac{mi^3}{\sec^2}$, the total energy is given by

$$\begin{aligned} H &= \frac{1}{2}(v(0))^2 - \frac{k}{r(0)} \\ &= \frac{9}{2} - \frac{95194.14}{5000} \\ &= 4.5 - 19.039 \\ &= -14.539 \end{aligned}$$

Moreover, $H < 0$ implies that the satellite is in a closed orbit—i.e., its orbit is an ellipse.

EXAMPLE 5 Suppose a satellite moving in a circle with radius R at a constant speed has a period of T. What is its energy integral?

Solution: To begin with, the position vector of the satellite is

$$\mathbf{r}(t) = R\left\langle \cos\left(\frac{2\pi t}{T}\right), \sin\left(\frac{2\pi t}{T}\right)\right\rangle$$

Thus, the satellite's velocity and speed are

$$\begin{aligned} \mathbf{v}(t) &= \frac{2\pi R}{T}\left\langle -\sin\left(\frac{2\pi t}{T}\right), \cos\left(\frac{2\pi t}{T}\right)\right\rangle \\ v &= \|\mathbf{v}\| = \frac{2\pi R}{T} \end{aligned}$$

Since $r = R$, the energy integral is

$$H = \frac{1}{2}\left(\frac{2\pi R}{T}\right)^2 - \frac{k}{R} = \frac{2\pi^2 R^2}{T^2} - \frac{k}{R}$$

The result in example 5 can be greatly improved. In particular, the acceleration of the satellite is

$$\mathbf{r}'' = R\left(\frac{2\pi}{T}\right)^2\left\langle -\cos\left(\frac{2\pi t}{T}\right), -\sin\left(\frac{2\pi t}{T}\right)\right\rangle, \qquad \|\mathbf{r}''\| = R\left(\frac{2\pi}{T}\right)^2$$

Keplers' problem says that

$$\|\mathbf{r}''\| = \frac{k}{R^3}\|\mathbf{r}\| \quad \Longrightarrow \quad R\left(\frac{2\pi}{T}\right)^2 = \frac{k}{R^2}$$

Solving for the period T then yields

$$T^2 = \frac{4\pi^2}{k}\ R^3$$

Substitution into H in example 5 yields

$$H = \frac{2\pi^2 R^2}{\frac{4\pi^2}{k}R^3} - \frac{k}{R} = \frac{-k}{2R}$$

Moreover, the speed v in example 5 is

$$v = \frac{2\pi R}{\sqrt{\frac{4\pi^2}{k}R^3}} = \sqrt{\frac{k}{R}}$$

Indeed, the results of example 5 can be extended into the following theorem:

Theorem 1: In order for a satellite to orbit the earth at a distance R from its center, it is necessary that the initial velocity $\mathbf{v}(0)$ be perpendicular to $\mathbf{r}(0)$, that $\|\mathbf{r}(0)\| = R$, and that

$$\|\mathbf{v}(0)\| = \sqrt{\frac{k}{R}}$$

In that case, the Hamiltonian energy will be $H = \frac{-k}{2R}$.

Check your Reading: Why is it necessary for $\mathbf{v}(0)$ to be perpendicular to $\mathbf{r}(0)$ in theorem 1?

Black Holes

Finally, let us notice that we can use conservation of energy to study more exotic objects in space, such as a black hole. Indeed, let us conclude by using the law of conservation of energy to determine to what size the earth would have to collapse in order for it to become a black hole.

To begin with, $H = 0$ is known as the **minimum escape energy** since $H < 0$ means no escape and $H > 0$ implies escape. Since light travels at a speed of $c = 186,000$ miles per second, we must determine the radius r such that light at velocity c corresponds to the minimum escape energy. That is, we must solve for r when $H = 0$ and $v = c$:

$$0 = \frac{1}{2}c^2 - \frac{k}{r}$$

The result is $r = 2k/c^2$, which means that the radius of the earth would have to be

$$r = \frac{2k}{c^2} = \frac{2 \cdot (95,194.14)}{(186,000)^2} = 5.5032 \times 10^{-6}\ miles$$

Indeed, converting to feet and then inches yields

$$r = 5.5032 \times 10^{-6} miles \times 5280\frac{ft}{mi} \times 12\frac{in}{ft} = 0.34868\ inches$$

That is, the entire earth would have to be compressed into a sphere which is a little less that 3/8 of an inch in order for it to form a black hole.

Exercises

Use the given initial conditions to find $\mathbf{L}$, *the equation of the plane containing the orbit, and the total energy* H *of the orbit. Does the satellite remain in orbit about the earth, or will it eventually escape the earth's gravitational field?*

1. $\mathbf{r}(0) = \langle 1,2,1\rangle$, $\mathbf{v}(0) = \langle 2,2,1\rangle$
2. $\mathbf{r}(0) = \langle 1,3,2\rangle$, $\mathbf{v}(0) = \langle 1,2,1\rangle$
3. $\mathbf{r}(0) = \langle 2,2,1\rangle$, $\mathbf{v}(0) = \langle 0,0,2\rangle$
4. $\mathbf{r}(0) = \langle 1,0,0\rangle$, $\mathbf{v}(0) = \langle 0,1,1\rangle$
5. $\mathbf{r}(0) = \langle 1,0,0\rangle$, $\mathbf{v}(0) = \langle 0,1,0\rangle$
6. $\mathbf{r}(0) = \langle 1,0,0\rangle$, $\mathbf{v}(0) = \langle 0,0,1\rangle$
7. $\mathbf{r}(0) = \langle 4063,0,0\rangle$, $\mathbf{v}(0) = \langle 0,4.8,0\rangle$
8. $\mathbf{r}(0) = \langle 4063,0,0\rangle$, $\mathbf{v}(0) = \langle 0,5,0\rangle$
9. $\mathbf{r}(0) = \langle 4063,0,0\rangle$, $\mathbf{v}(0) = \langle 0,5.2,0\rangle$
10. $\mathbf{r}(0) = \langle 4063,0,0\rangle$, $\mathbf{v}(0) = \langle 7,0,0\rangle$
11. $\mathbf{r}(0) = \langle 5,5,2\rangle$, $\mathbf{v}(0) = \langle -5,-5,-2\rangle$
12. $\mathbf{r}(0) = \langle 1,0,0\rangle$, $\mathbf{v}(0) = \langle 0,0,0\rangle$

13. (**Energy Integral)** In this exercise, we show using a different method that

$$\frac{1}{2}v^2 - \frac{k}{r} = H$$

where H is a constant.

(a) Compute $\frac{d}{dt}\left(\frac{1}{2}v^2\right)$ using the fact that

$$v^2 = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

Then write the result as the dot product of two vectors.

(b) Compute $\frac{d}{dt}\frac{k}{r}$ using the fact that

$$\frac{k}{r} = \frac{k}{\sqrt{x^2 + y^2 + z^2}}$$

Then write the result as the dot product of two vectors.

(c) Combine the results in (a) and (b). What is the result? What does this say about H?

14. Show that $\phi(\mathbf{r}) = -k/r$ is a potential for

$$\mathbf{F}(x, y, z) = \left\langle \frac{-kx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-ky}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-kz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle$$

where $r = \sqrt{x^2 + y^2 + z^2}$.