LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2011, Prepared by Dr. Jeff Knisley
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NAME ___________________________ STUDENT NUMBER ___________________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The bold
faced numbers in parentheses indicate the number of the topics covered
in that problem from the Study Guide. No calculators!!! You may omit two
numbered problems. Indicate which two problems you are omitting: _____ and _____.

1. Find the solution set of the system $Ax = b$, where $x \in \mathbb{R}^3$,
   \[
   A = \begin{bmatrix}
   1 & 2 & 3 \\
   4 & 5 & 6 \\
   7 & 8 & 9 \\
   \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix}
   1 \\
   2 \\
   3 \\
   \end{bmatrix}
   \]
   and express the solution as the translation of a vector space. ($A3, A4, A5, D6$)

2. Transform the basis $\{ (1,1,0), (1,0,1), (1,1,1) \}$ for $\mathbb{R}^3$ into an orthogonal
   basis using the Gram-Schmidt process. ($C17, C19, C20, C21$)

3. (a) What is an elementary matrix?
   (b) Express $A$ as a product of elementary matrices where
   \[
   A = \begin{bmatrix}
   4 & 3 & 0 \\
   -4 & 0 & 1 \\
   1 & 1 & 0 \\
   \end{bmatrix}
   \]
   Identify each elementary matrix in the product. ($D7$)

4. Let $V$ denote the space of all functions of the form
   \[
   p(x) = ax^{-1} + b + cx
   \]
   (that is, $V = \{ ax^{-1} + b + cx : a, b, c \in \mathbb{R} \}$ where $x \neq 0$ is a variable).
   Define a transformation by
   \[
   (Tp)(x) = p(x) + p(x^{-1})
   \]
   Show that $T$ is a linear transformation on $V$. Is $T$ invertible? Explain. ($C7, C9, C10$).
5. Give an example of a $3 \times 3$ antisymmetric matrix – i.e., a matrix $A$ for which

$$A^T = -A$$

Prove that if $A$ is an anti-symmetric $3 \times 3$ matrix, then $(Ax) \cdot x = 0$ for all $x \in \mathbb{R}^3$. (B8, D4).

6. State the definition of vector space. (C1)

7. Prove the following: If $A$ is an upper triangular $5 \times 5$ matrix with zeros on the diagonal, then $A^5$ is the zero matrix. (D1, D2, D11)

8. Let $V$ be the 3-dimensional vector space of linear combinations of sine and cosine functions in the form

$$V = \{a + b \cos(\theta) + c \sin(\theta) \mid a, b, c \in \mathbb{R}\}$$

Find the matrix $A$ that represents the linear transformation $\frac{d}{d\theta}$ on $V$ relative to the ordered basis $\{1, \cos(\theta), \sin(\theta)\}$. (C7, C8)

9. Find the eigenvalues (they are integers) and the eigenvectors of (A9, D14, D17, D18, D19):

$$A = \begin{bmatrix}
5 & 6 & 0 & 0 \\
-2 & -2 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & 2 & 5
\end{bmatrix}.$$

10. Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{R}$ and suppose that the eigenvectors of $A$ form a linearly independent set $\{v_1, \ldots, v_n\}$ over $\mathbb{R}^n$. Prove that if each $v_j$, $j = 1, \ldots, n$ is also an eigenvector of $B$, then $AB = BA$. (D17, D19, D23)