LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2010, Prepared by Dr. Jeff Knisley

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NAME ______________________________ STUDENT NUMBER _______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Determine the solution of the system

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & x_3 & - & x_4 & = & 5 \\ x_1 & + & x_2 & - & x_3 & + & x_4 & = & 3 \\ 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & = & 2 \\ 3x_1 & + & 7x_2 & - & 2x_3 & - & 3x_4 & = & -1 \end{array}$$

as the translation of a vector space. **(A1, A2, A3, A7, A9, B4)**

2. State three conditions on an $n \times n$ matrix A which would (each) imply that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Does the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \langle 1, 0, 0 \rangle^t$ and

$$A = \begin{bmatrix} 11 & 0 & 4 \\ -8 & 3 & -4 \\ -4 & 0 & 1 \end{bmatrix},$$

have a unique solution (explain)? (**A5, A8, A9**)

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $[1, 2, 3]$ and $[4, 5, 6]$ and passes through the point $(7, 8, 9)$. Find the equation of the plane (in terms of x, y, and z coordinates) and express the plane as a translation of a vector space. **(B4, B12)**

4. Show that $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less, is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism and verify that it is an isomorphism). **(C12, C13, C15)**

5. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain. **(C5, C11, C15)**

6. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T((x_1, x_2, x_3)) = (x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3 + x_4). \qquad \textbf{(C7, C8)}$$

7. Find the orthogonal complement of span$\{[-1, 2, 0, 3], [0, 4, 1, -2]\}$ in $\mathbb{R}^4$. **(C3, C18)**

8. Transform the basis $\{(1, 0, 1), (0, 1, 2), (2, 1, 0)\}$ for $\mathbb{R}^3$ into an orthonormal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

9. Find the eigenvalues (they are integers) and the eigenvectors of:

$$A = \begin{bmatrix} -2 & 0 & 3 \\ -5 & -2 & -3 \\ 2 & 0 & 3 \end{bmatrix}.$$

(A9, D14, D17, D18, D19)

10. Let $\mathbf{v}$ be a column vector in $\mathbb{R}^n$ and let $\mathbf{v}^t$ be the transpose of $\mathbf{v}$. What are the eigenvalues and associated eigenspaces of the matrix

$$A = \mathbf{v}\,\mathbf{v}^t$$

given that $\mathbf{v} \neq \mathbf{0}$. **(D1, D17, D18, D20, D22)**