LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2011, Prepared by Dr. Jeff Knisley

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NAME ________________________________ STUDENT NUMBER _________________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution to this system of equations:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & 2 \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & 4 \\ 3x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 5 \\ -5x_1 & - & 2x_2 & - & x_3 & + & x_4 & = & 5 \end{array}$$

   (**A1, A2, A3, A7, B4**)

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Does the system

$$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 5 & 7 \\ 1 & 11 & 16 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 31 \\ 70 \end{bmatrix}$$

   have a unique solution (explain)? (**A5, A8, A9**)

3. State the definition of *vector space*. (**C1**)

4. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T((x_1, x_2, x_3)) = (x_1 - x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3).$$

   (**C7, C8**)

5. Consider the space $\mathcal{P}_3$ of all polynomials of degree 3 or less. Find a matrix representation of the derivative operator with respect to the basis $\{ 1, x, x^2, x^3 \}$. (**C11, C6, A1**).

6. Transform the basis

$$\{\langle 1, 0, 1, 0\rangle, \langle 1, 0, -1, 0\rangle, \langle 0, 1, 0, 1\rangle, \langle 0, -1, 0, 1\rangle\}$$

   for $\mathbb{R}^4$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

7. Express A and A^{-1} as products of elementary matrices where

$$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}.$$

(**D3, D7, D8, D9**)

8. Show that if A is an orthogonal $n \times n$ matrix, i.e., if $A^T = A^{-1}$, then the columns of A form an orthonormal basis of $\mathbb{R}^n$. (**D6, D10**)

9. Find the eigenvalues (they are integers) and the eigenvectors of

$$A = \begin{bmatrix} -1 & 0 & 2 \\ -2 & 1 & 2 \\ -4 & 0 & 5 \end{bmatrix}.$$

What are the dimensions of the eigenspaces? (**A9, D14, D17, D18, D19**)

10. Consider

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}.$$

Put A in a row echelon form, keeping track of how each row operation affects the determinant. Then calculate the determinant of A. **(A4, D15)**