LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2010, Prepared by Dr. Jeff Knisley

February 12, 2010

NAME ______________________________ STUDENT NUMBER _______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Consider the matrix

$$A = \begin{bmatrix} -1 & 5 & -1 & 1 & 2 \\ 2 & -10 & 2 & -3 & 5 \\ -2 & 9 & -4 & 1 & 7 \\ 0 & 2 & 0 & 2 & 3 \end{bmatrix}$$

   Put A in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. (**A3, A4, A5**)

2. State three conditions on an $n \times n$ matrix A which would (each) imply that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Does the system $A\mathbf{x} = \mathbf{0}$ (where $\mathbf{0}$ is the zero vector) given the matrix

$$A = \begin{bmatrix} 11 & 0 & 4 \\ -8 & 3 & -4 \\ -4 & 0 & 1 \end{bmatrix}$$

   have a unique solution (explain)? (**A5, A8, A9**)

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $\langle 1, 0, 1\rangle$ and $\langle 2, 3, 0\rangle$ and passes through the point $(3, 3, 1)$. Find the equation of the plane (in terms of x, y, and z coordinates) and express the plane as a translation of a vector space. (**B4, B12**)

4. State the "Fundamental Theorem of Finite Dimensional Vector Spaces" (which deals with isomorphisms of vector spaces). Let V_1 and V_2 be vector spaces with real scalars. What conditions must be satisfied for $\pi : V_1 \to V_2$ to be an isomorphism? Are the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ isomorphic? Explain (**B4, B12).**

5. Let $\mathbf{v}_1 = 1 + x - x^3$, $\mathbf{v}_2 = x^2 + 2$, and $\mathbf{v}_3 = x^3 - 1$ be vectors in $\mathcal{P}_3$, the vector space of all polynomials with real coefficients of degree 3 or less. Are these vectors linearly independent? Can every vector in $\mathcal{P}_3$ be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? Explain. (**C5, C6, C11, C15**).

6. Find the projection of x onto $\sin(x)$ in the inner product space $C_{[-\pi,\pi]}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

(**C15, C17**)

7. Express A and A^{-1} as products of elementary matrices, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(**D3, D7, D8, D9**)

8. Prove the following: If $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and if $m > n$, then A cannot be invertible. (**D6, D10**)

9. Prove the following: If U is an invertible $n \times n$ matrix for which $U^T = -U^{-1}$, then any eigenvalue of U has a magnitude of 1.

10. Diagonalize $A = \begin{bmatrix} 6 & 10 \\ -2 & -3 \end{bmatrix}$ and use the diagonalization to calculate A^{10}. (**D1, D17, D20**):