LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2010, Prepared by Dr. Jeff Knisley
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NAME ________________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Solve the system of equations (**A2, A6, A7**):

$$\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & - & 7x_3 & = & -3 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 7 \\ 4x_1 & + & 5x_2 & - & 2x_3 & = & 10 \end{array}$$

2. Prove that if $\vec{x}_1$ and $\vec{x}_2$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution. (**A9, C7**)

3. Use Schwarz's Inequality, which states that

$$|\vec{v}_1 \cdot \vec{v}_2| \;\leq \|\vec{v}_1\| \; \|\vec{v}_2\| \;\; for\ all \;\; \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$$

 to prove the Triangle Inequality in $\mathbb{R}^n$. (**B8, B10**)

4. Show that the vector space

$$V = \{m + a\cos(x) + b\sin(x) \;\; | \;\; m, a, b \in \mathbb{R}\}$$

 is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism). (**C12, C13, C15**)

5. Find the projection of x onto $\sin x$ in the inner product space $C_{0,\pi}$ of continuous functions on $[0, \pi]$ with the inner product of f and g defined as (**C15, C17**)

$$\langle f, g \rangle = \int_0^{\pi} f(x) g(x)\, dx.$$

6. State three conditions on an $n \times n$ matrix A which would (each) imply that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Does the system $A\mathbf{x} = \mathbf{0}$ (where $\mathbf{0}$ is the zero vector) given the matrix

$$A = \begin{bmatrix} 11 & 0 & 4 \\ -8 & 3 & -4 \\ -4 & 0 & 1 \end{bmatrix}$$

have a unique solution (explain)? (**A5, A8, A9**)

7. Transform the basis $\{\langle 1,0,1\rangle, \langle 1,1,1\rangle, \langle 0,1,2\rangle\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

8. Find the rank, nullity, and a basis for the column space of (**A3, 4, A5, D6, D10**):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -6 & -1 \\ 3 & -10 & -7 \end{bmatrix}.$$

9. Find the eigenvalues (they are integers) and the eigenvectors of (**A9, D14, D17, D18, D19**):

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

10. Prove the following: If A is a symmetric $n \times n$ matrix, then the eigenvalues of A must be real numbers (**D14, D17, D18, D19**).