LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2009, Prepared by Dr. Jeff Knisley

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NAME ________________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} \in R^3$,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   and express the solution as the translation of a vector space. (**A3, A4, A5, D6**)

2. Transform the basis $\{\langle 1,1,0\rangle, \langle 1,0,1\rangle, \langle 1,1,1\rangle\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

3. (a) What is an elementary matrix?
   (b) Express A as a product of elementary matrices where

$$A = \begin{bmatrix} 4 & 3 & 0 \\ -4 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

   Identify each elementary matrix in the product. (**D7)**

4. Let V denote the space of all functions of the form

$$p(x) = ax^{-1} + b + cx$$

   (that is, $V = \{ax^{-1} + b + cx \;:\; a, b, c \in \mathbb{R}\}$ where $x \neq 0$ is a variable). Define a transformation by

$$(Tp)(x) = p(x) + p\left(x^{-1}\right)$$

   Show that T is a linear transformation on V. Is T invertible? Explain. (**C7, C9, C10**).

5. Give an example of a 3×3 *antisymmetric matrix* – i.e., a matrix A for which

$$A^T = -A$$

Prove that if A is an anti-symmetric 3×3 matrix, then $(A\mathbf{x}) \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^3$. (**B8, D4**).

6. State the definition of vector space. (**C1**)

7. Prove the following: If A is an upper triangular 5×5 matrix with zeros on the diagonal, then A^5 is the zero matrix. (**D1, D2, D11**)

8. Let V be the 3-dimensional vector space of linear combinations of sine and cosine functions in the form

$$V = \{a + b\cos(\theta) + c\sin(\theta) \quad | \quad a, b, c \in \mathbb{R}\}$$

Find the matrix A that represents the linear transformation $\frac{d}{d\theta}$ on V relative to the ordered basis $\{1, \cos(\theta), \sin(\theta)\}$. (**C7, C8**)

9. Find the eigenvalues (they are integers) and the eigenvectors of (**A9, D14, D17, D18, D19**):

$$A = \begin{bmatrix} 5 & 6 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 5 \end{bmatrix}.$$

10. Let A and B be $n \times n$ matrices over $\mathbb{R}$ and suppose that the eigenvectors of A form a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ over $\mathbb{R}^n$. Prove that if each $\mathbf{v}_j$, $j = 1, \ldots, n$ is also an eigenvector of B, then $AB = BA$. (**D17, D19, D23**)