LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2009, Prepared by Dr. Jeff Knisley

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Consider the matrix
$$A = \begin{bmatrix} 0 & 2 & 3 & 2 \\ 1 & -2 & 0 & 0 \\ 1 & -2 & 0 & 1 \\ 0 & 2 & 3 & -2 \end{bmatrix}$$
Put A in row echelon form. At each step, tell the elementary row operation you are using. What is the dimension of the column space of A? (**A3, A4, A5, D6**)

2. Find all solutions to the following system of equations
$$\begin{array}{llllll} x_1 & -3x_2 & +3x_3 & +3x_4 & & =1 \\ x_1 & -2x_2 & +x_3 & +x_4 & & =0 \\ x_1 & -2x_2 & & +x_4 & +3x_5 & =0 \\ x_1 & -2x_2 & -x_3 & & +2x_5 & =0 \\ x_1 & & +x_3 & +x_4 & +3x_5 & =0 \end{array}$$
(**A1, A5, A8, A9**)

3. State the definition of *vector space.* (**C1**)

4. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T((x_1, x_2, x_3)) = (x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3 + x_4)$. (**C7, C8**)

5. Consider the space $\mathcal{P}_3$ of all polynomials of degree 3 or less. Define a transformation by

$$(Tp)(x) = \begin{cases} \frac{p(x)-p(0)}{x} & if \quad x \neq 0 \\ p'(0) & if \quad x = 0 \end{cases}$$

where $p'(x)$ is the derivative of p with respect to x. Show that T is a linear transformation on $\mathcal{P}_3$. Explain why T is not invertible. (**A1, C1, C6, C7, D8**).

6. Transform the basis $\{\langle 1,1,0\rangle, \langle 0,1,0\rangle, \langle 0,1,1\rangle\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

7. Prove the following: A linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if 0 is not an eigenvalue of A. (**D3, D8, D9, D17**)

8. Prove the following: If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation whose column space has a dimension n, then its nullspace is trivial. (**D6, D10**)

9. Find the eigenvalues (they are integers) and the eigenvectors of (**A9, D14, D17, D18, D19**):

$$A = \begin{bmatrix} -8 & 0 & -12 \\ 4 & 0 & 6 \\ 7 & -1 & 11 \end{bmatrix}.$$

10. Show that every diagonal entry on an upper triangular matrix is an eigenvalue of the matrix. (**D17, D19, D23**)