LINEAR ALGEBRA COMPREHENSIVE EXAM
Fall 2005, Prepared by Dr. Robert Gardner
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NAME ____________________________ STUDENT NUMBER ____________________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The bold
faced numbers in parentheses indicate the number of the topics covered in that problem
from the Study Guide. No calculators!!! You may omit two numbered problems. Indicate
which two problems you are omitting: ______ and ______.

1. Find the solution set of $A\vec{x} = \vec{b}$ where
   
   $A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$

   and express the solution as a translation of a vector space. (A1, A7, B4)

2. Give three conditions on $n \times n$ matrix $A$ which would (each) imply that the system
   $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that
   $A\vec{x} = \vec{b}$ has multiple solutions. (A5, A8, A9)

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$
   of continuous functions on $[0,2\pi]$ with the inner product of $f$ and $g$ defined as
   
   $\langle f, g \rangle = \int_{0}^{2\pi} f(x)g(x) \, dx$.

   (B8, B9, C15)

4. Find the projection of vector $[1, 2, 3, 4]$ onto the line joining the points $(0, 4, -3, 2)$ and
   $(1, 4, 0, 2)$. (B3, B7, B8, C17)

5. Find the standard matrix representation (i.e. the representation with respect to the
   standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by
   
   $T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3)$. (C7, C8)

6. Consider the space $P_3$ of all polynomials of degree 3 or less. Find the coordinate vector
   of $x^3 + 3x^2 - 4x + 2$ relative to the ordered basis $(x, x^2 - 1, x^3, 2x^2)$. (C11, C6, A1)

7. Transform the basis $\{(1, 0, 1), (0, 1, 2), (2, 1, 0)\}$ for $\mathbb{R}^3$ into an orthogonal basis using
   the Gram-Schmidt process. (C17, C19, C20, C21)

8. Prove that for $A = [a_{ij}]$ and $B = [b_{ij}]$ $n \times n$ matrices, we have $(AB)^T = B^T A^T$. (D1, D4)
9. Prove that if $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$, then the set $E_\lambda$ consisting of the zero vector together with all eigenvectors of $A$ for this eigenvalue $\lambda$ is a subspace of $n$-space. (C4, D17, D19)

10. Diagonalize $A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$ and calculate $A^{100}$. (D1, D17, D20)