LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2002, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of $A\vec{x} = \vec{b}$ where
$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -12 & 5 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 8 \\ 26 \\ 14 \end{bmatrix}.$$
(**A1, A7, B4**)

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that $A\vec{x} = \vec{b}$ has multiple solutions. (**A5, A8, A9**)

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$with the inner product of f and g defined as
$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$
(**B8, B9, C15**)

4. Find the projection of vector $[1, 2, 3, 4]$ onto the line joining the points $(0, 4, -3, 2)$ and $(1, 4, 0, 2)$. (**B3, B7, B8, C17**)

5. Show that $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less, is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism). (**C12, C13, C15**)

6. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3)$. (**C7, C8**)

7. Consider the space $\mathcal{P}_3$ of all polynomials of degree 3 or less. Find the coordinate vector of $x^3 + 3x^2 - 4x + 2$ relative to the ordered basis$(x, x^2 - 1, x^3, 2x^2)$. (**C11, C6, A1**).

8. Subspaces U and W of $\mathbb{R}^n$ are *orthogonal* if $\vec{u} \cdot \vec{w} = 0$ for all $\vec{u} \in U$ and all $\vec{w} \in W$. Let U and W be orthogonal subspaces of $\mathbb{R}^n$, and let $\dim(U) = n - \dim(W)$. Show that each subspace is the orthogonal complement of the other. (**C18**)

9. Transform the basis $\{(1,0,1),(0,1,2),(2,1,0)\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

10. Prove that for $A = [a_{ij}]$ and $B = [b_{ij}]$ $n \times n$ matrices, we have $(AB)^T = B^T A^T$. (**D1, D4**)

11. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)

12. Diagonalize $A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$ and calculate A^{100}. (**D1, D17, D20**)