LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2021b, Prepared by Dr. Robert Gardner

November 19, 2021

NAME ______________________________ Start Time: ________ End Time: ________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Find the solution of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ -6 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. **(A1, A7, B4)**

2. Explain the difference between a *vector* in $\mathbb{R}^n$ and a *point* in $\mathbb{R}^n$. (**B1, B3**)

3. Consider the inner product space $C_{-\pi,\pi}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

In the vector space $C_{-\pi,\pi}$ find the angle between $\cos x$ and $\sin x$. **(B8, B9, C15)**

4. State the definition of *vector space*. **(C1)**

5. Consider the vectors $\vec{v}_1 = x^2 - 2x + 1$, $\vec{v}_2 = 2x^2 + 5x + 11$, and $\vec{v}_3 = 3x^2 + 7x + 17$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. If you are not using ordered bases then you are not arguing correctly!!! **(C5, C11, C15)**

6. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = D(p(x))$, the derivative of $p(x)$. Let the ordered basis for $\mathcal{P}_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix A which represents T relative to B, B'. **(C7, C8, C11, C15)**

7. Transform the basis $\{[1,1,0],[0,1,2],[1,1,1]\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. **(C17, C19, C20, C21)**

8. Consider

$$A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

Find the rank and a basis for the row space. (**A4, A5, D6, D10**)

9. Find the eigenvalues (they are integers):

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

(A9, D14, D17, D18, D19)

10. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)