Be clear and give all details. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. 

No calculators and turn off your cell phones! You may omit two problems. Indicate which two problems you are omitting: _____ and _____ . There is a three hour time limit.

1. Show that if the system of equations $A\vec{x} = \vec{b}$ has two distinct solutions, then it has an infinite number of solutions. (A1, A6, A8)

2. Let $A$ be an $m \times n$ matrix. Show that $\{ \vec{x} \mid A\vec{x} = \vec{0} \}$ is a subspace of $\mathbb{R}^n$. (A8, A9, C4)

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$ with the inner product of $f$ and $g$ defined as

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) \, dx.$$ 

(B8, B9, C15)

4. State the definition of vector space. (C1)

5. Consider the vectors $\vec{v}_1 = x^2 - 2x + 1$, $\vec{v}_2 = 2x^2 + 5x + 11$, and $\vec{v}_3 = 3x^2 + 7x + 17$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. If you are not using ordered bases then you are not arguing correctly!!! (C5, C11, C15)

6. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (C17, C19, C20, C21)

7. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle \vec{v}, \vec{w} \rangle| \leq ||\vec{v}||||\vec{w}||$, to prove the Triangle Inequality in an inner-product space. (B8, B10, C15)

8. Consider

$$A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$ 

Find the rank, a basis for the row space, and a basis for the column space. (A4, A5, D6, D10)
9. Prove that if \( \lambda \) is an eigenvalue of an \( n \times n \) matrix \( A \), then the set \( E_\lambda \) consisting of the zero vector together with all eigenvectors of \( A \) for this eigenvalue \( \lambda \) is a subspace of \( n \)-space. (C4, D17, D19)

10. Diagonalize

\[
A = \begin{bmatrix}
7 & 8 \\
-4 & -5
\end{bmatrix}
\]

and find \( A^5 \). (D1, D17, D18, D20, D22)