LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2018a, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Show that if the system of equations $A\vec{x} = \vec{b}$ has two distinct solutions, then it has an infinite number of solutions. **(A1, A6, A8)**

2. Let A be an $m \times n$ matrix. Show that $\{\vec{x} \mid A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$. **(A8, A9, C4)**

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$with the inner product of f and g defined as

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$

(**B8, B9, C15**)

4. State the definition of *vector space*. **(C1)**

5. Consider the vectors $\vec{v}_1 = x^2 - 2x + 1$, $\vec{v}_2 = 2x^2 + 5x + 11$, and $\vec{v}_3 = 3x^2 + 7x + 17$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and **ordered bases**. If you are not using ordered bases then you are not arguing correctly!!! **(C5, C11, C15)**

6. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2],$ $[2, 1, 1, 1],$ $[1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

7. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle\vec{v}, \vec{w}\rangle| \leq \|\vec{v}\|\|\vec{w}\|$, to prove the Triangle Inequality in an inner-product space. **(B8, B10, C15)**

8. Consider

$$A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

Find the rank, a basis for the row space, and a basis for the column space. (**A4, A5, D6, D10**)

9. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)

10. Diagonalize

$$A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$$

and find A^5. **(D1, D17, D18, D20, D22)**