LINEAR ALGEBRA COMPREHENSIVE EXAM
Fall 2016a, Prepared by Dr. Robert Gardner
September 9, 2016

NAME ___________________________ Start Time: _______ End Time: _______

Be clear and give all details. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators and turn off your cell phones! You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Express the solution of this system as a translation of a vector space:

\[
\begin{align*}
3x_1 + 2x_2 + x_3 &= 4 \\
-5x_1 - 3x_2 - x_3 + x_4 &= -7 \\
5x_1 + 4x_2 + 3x_3 + 2x_4 &= 6 \\
x_1 + x_2 + x_3 + x_4 &= 1
\end{align*}
\]

(A1, A2, A3, A7, B4)

2. Let \( A \) be an \( m \times n \) matrix. Show that \( \{ \vec{x} | A\vec{x} = \vec{0} \} \) is a subspace of \( \mathbb{R}^n \). (A8, A9, C4)

3. Consider

\[
A = \begin{bmatrix}
0 & 6 & 6 & 3 \\
1 & 2 & 1 & 1 \\
4 & 1 & -3 & 4 \\
1 & 3 & 2 & 0
\end{bmatrix}
\]

Find the rank, a basis for the row space, and a basis for the column space. (A4, A5, D6, D10)

4. Use the Schwarz Inequality, which states that for vectors \( \vec{v} \) and \( \vec{w} \) in an inner-product space, we have \( |\langle \vec{v}, \vec{w} \rangle| \leq ||\vec{v}|| ||\vec{w}|| \), to prove the Triangle Inequality in an inner-product space. (B8, B10, C15)

5. State the definition of vector space. (C1)

6. Consider the vectors \( \vec{v}_1 = x^2 - 2x + 1 \), \( \vec{v}_2 = 2x^2 + 5x + 11 \), and \( \vec{v}_3 = 3x^2 + 7x + 17 \) in \( P_2 \), the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. (C5, C11, C15)
7. Let \( T: \mathcal{P}_3 \rightarrow \mathcal{P}_3 \) be defined by \( T(p(x)) = D(p(x)) \), the derivative of \( p(x) \). Let the ordered basis for \( \mathcal{P}_3 \) be \( B = B' = (x^3, x^2, x, 1) \). Find the matrix \( A \) which represents \( T \) relative to \( B, B' \). (C7, C8, C11, C15)

8. Transform the basis \( \{(1,0,1),(0,1,2),(2,1,0)\} \) for \( \mathbb{R}^3 \) into an orthogonal basis using the Gram-Schmidt process. (C17, C19, C20, C21)

9. Find two of the eigenvalues (they are integers) and two of the eigenvectors for \( \{A9, D14, D17, D18, D19\} \):

\[
A = \begin{bmatrix}
-2 & 0 & 0 \\
-5 & -2 & -5 \\
5 & 0 & 3
\end{bmatrix}
\]

10. Diagonalize \( A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix} \) and calculate \( A^{100} \). Of course, you may leave your answer in terms of powers of certain numbers. (D1, D17, D20)