LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2015a, Prepared by Dr. Robert Gardner

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NAME ______________________________ Start Time: ________ End Time: ________

Be clear and **give all details**. Use symbols correctly (such as equal signs) and write in complete sentences. The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side.** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 14 & -4 & 7 \\ -3 & -6 & 0 & -9 \\ 5 & -8 & 6 & 9 \\ 2 & 13 & -3 & 9 \end{bmatrix}.$$

Which columns contain pivots? What is a basis for the column space? **(A3, A4, A5, D6)**

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that $A\vec{x} = \vec{b}$ has multiple solutions. (**A5, A8, A9**)

3. Prove that if $\vec{x}_1$ and $\vec{x}_2$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution. **(A9, C7)**

4. Find the projection of $[1, 2, 1, 2]$ onto the plane $x + y + z + w = 0$. Explain your reasoning! **(B3, B7, B8, C17, C19)**

5. State the definition of *vector space*. **(C1)**

6. Consider the vectors $\vec{v}_1 = x^2 - 2x + 1$, $\vec{v}_2 = 2x^2 + 5x + 11$, and $\vec{v}_3 = 3x^2 + 7x + 17$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. **(C5, C11, C15)**

7. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2],$ $[2, 1, 1, 1]$, $[1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

8. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$, to prove the Triangle Inequality in an inner-product space. Here, $\langle \vec{v}, \vec{w} \rangle$ denotes the inner product. **(B8, B10, C15)**

9. Diagonalize $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$. (**D2, D17, D18, D19, D20**)

10. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)