LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2008, Prepared by Dr. Robert Gardner

September 19, 2008

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. (**A1, A7, B4**)

2. State three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Does the system

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

have a unique solution (explain)? (**A5, 8, A9**)

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $[1, 2, 3]$ and $[4, 5, 6]$ and passes through the point $(7, 8, 9)$. Find the equation of the plane (in terms of x, y and z coordinates) and express the plane as a translation of a vector space. (**B4, B12**)

4. State the "Fundamental Theorem of Finite Dimensional Vector Spaces" (which deals with isomorphisms of vector spaces). Let V_1 and V_2 be vector spaces with real scalars. What conditions must be satisfied for $\pi : V_1 \to V_2$ to be an isomorphism? Are the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ isomorphic? Explain. (**B4, B12**)

5. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$with the inner product of f and g defined as

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\, dx.$$

(**B8, B9, C15**)

6. Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Show that there is one and only one vector $\vec{p}$ in W such that $\vec{b} - \vec{p}$ is perpendicular to every vector in W. **(C4, C18, C19)**

7. Let A be an $n \times n$ matrix. Prove that the collection of all solutions to the equation $A\vec{x} = \vec{0}$ form a subspace of $\mathbb{R}^n$. (**A9, C2, C4**)

8. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

9. (a) What is an elementary matrix? **(D7)**

(b) Express A and A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$. **(D3, D7, D8, D9)**

10. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. What is the eigenspace? **(D12, D17, D18, D19)**