

LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2007, Prepared by Dr. Robert Gardner

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NAME _____ STUDENT NUMBER _____

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: _____ and _____.

1. Find the solution set of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. (**A1, A7, B4**)

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that $A\vec{x} = \vec{b}$ has multiple solutions. (**A5, A8, A9**)

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$ with the inner product of f and g defined as

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) dx.$$

(**B8, B9, C15**)

4. Find the projection of vector $[1, 2, 3, 4]$ onto the line joining the points $(0, 4, -3, 2)$ and $(1, 4, 0, 2)$. (**B3, B7, B8, C17**)

5. Find the standard matrix representation (i.e. the representation with respect to the standard bases of \mathbb{R}^m and \mathbb{R}^n) of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by $T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3)$. (**C7, C8**)

6. Do each of the following:

(a) What is the vector space \mathbb{R}^n ? (**B1**)

(b) What is the difference between a point in \mathbb{R}^n and a vector in \mathbb{R}^n ? (**B3**)

(c) What is a vector space isomorphism? (**C13**)

7. Consider the vector spaces $V = V' = \text{span}\{\cos x, \sin x\}$ with ordered bases $B = B' = \{\cos x, \sin x\}$. Let $T : V \rightarrow V'$ be defined as the differentiation operator. Find the matrix A that represents T relative to B, B' . (C7, C8, C11, C15)
8. Let W be a subspace of \mathbb{R}^n and let \vec{b} be a vector in \mathbb{R}^n . Show that there is one and only one vector \vec{p} in W such that $\vec{b} - \vec{p}$ is perpendicular to every vector in W . (C4, C18, C19)
9. Diagonalize $\begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$ and find A^{100} . (D1, D17, D18, D20, D22)
10. (a) What is an elementary matrix? (D7)
- (b) Express A and A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$. (D3, D7, D8, D9)