LINEAR ALGEBRA COMPREHENSIVE EXAM

Fall 2002, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Solve the system
$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 4 \\ 2x_1 & - & 3x_2 & + & 2x_3 & - & x_4 & = & -1 \\ 3x_1 & - & 5x_2 & + & 3x_3 & - & 4x_4 & = & 3 \\ -x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 5 \end{array}.$$
(**A1, A2, A3, A6, A7**)

2. Prove that if $\vec{x_1}$ and $\vec{x_2}$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x_1}$ and $\vec{x_2}$ is also a solution. (**A9**)

3. Find the coordinate vector of $2x^2 + 3x - 1$ relative to the ordered basis $(x^2 - x, 2x^2 - 2x + 1, x^2 - 2x)$. (**A1, C6, C11**)

4. Find a unit vector in $\mathbb{R}^3$ which is orthogonal to both $\vec{x} = (1, 2, 3)$ and $\vec{y} = (0, 1, 1)$. (**B8, B9, B12**)

5. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, x_1)$. (**C7, C8**)

6. State the definition of *vector space*. (**C1**)

7. Use Schwarz's Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space we have $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$, to prove the Triangle Inequality in an inner-product space. (**B8, B10, C15**)

8. Let A be an $n \times n$ matrix. Prove that the collection of all solutions to the equation $A\vec{x} = \vec{0}$ form a subspace of $\mathbb{R}^n$ (**A9, C2, C4**)

9. Express A and A^{-1} as products of elementary matrices where
$$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}.$$
(**D3, D7, D8, D9**)

10. Consider

$$A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

Find the rank, a basis for the row space, and a basis for the column space.

11. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)

12. Consider

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}.$$

Put A in a row echelon form, keeping track of how each row operation affects the determinant. Then calculate the determinant of A. **(A4, D15)**