LINEAR ALGEBRA COMPREHENSIVE EXAM

Summer 2005, Prepared by Dr. Robert Gardner

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NAME________________________ STUDENT NUMBER____________________________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. Indicate which two problems you are omitting: ______ and ______. There is a three hour time limit. **No calculators!**

1. Express the solution of this system as a translation of a vector space **(A1, A2, A3, A7, B4)**:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & & & = & 5 \\ -5x_1 & - & 3x_2 & - & x_3 & + & x_4 & = & -7 \\ 5x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \end{array}$$

2. Prove that if $\vec{x}_1$ and $\vec{x}_2$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution. **(A9, C7)**

3. Find the part of vector $[-2, 4, 1, 0]$ which is perpendicular to plane $x + y + z + w = 5$. **(B3, B7, B8, C17, C19)**

4. Consider the inner product space $C_{-\pi,\pi}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

In the vector space $C_{-\pi,\pi}$, find the angle between $\cos x$ and $\sin x$. **(B8, B9, C15)**

5. State the definition of *vector space*. **(C1)**

6. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = D(p(x))$, where D is the differentiation operator. Let the ordered basis for $\mathcal{P}_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix A which represents T relative to B, B'. **(C7, C8, C11, C15)**

7. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain. **(C5, C11, C15)**

8. Consider

$$\begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

Find the rank, a basis for the row space, and a basis for the column space. **(A4, A5, D6, D10)**

9. (a) What is an elementary matrix? **(D7)**

(b) Express A and A^{-1} as products of elementary matrices where $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$. **(D3, D7, D8, D9)**

10. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. **(C4, D17, D19)**