LINEAR ALGEBRA COMPREHENSIVE EXAM

Summer 2004, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & -1 & 3 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix}.$$

Put A in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. **(A3, A4, A5)**

2. Let A be an $m \times n$ matrix. Show that $\{\vec{x} \mid A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$. **(A8, A9, C4)**

3. Consider the plane in $\mathbb{R}^3$ which contains the three points $(1, 0, 0)$, $(0, 1, -1)$, and $(1, 1, 1)$. Find the equation of the plane (in terms of x, y, and z coordinates) and express the plane as a translation of a vector space. **(B4, B12)**

4. State the definition of *vector space*. **(C1)**

5. Show that $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less, is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism and verify that it is an isomorphism). **(C12, C13, C15)**

6. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain. **(C5, C11, C15)**

7. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$, to prove the Triangle Inequality in an inner-product space. **(B8, B10, C15)**

8. Find the rank, nullity, and a basis for the column space of **(A9, D14, D17, D18, D19)**:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -4 & 2 \\ 3 & -10 & -7 \end{bmatrix}.$$

9. Find the eigenvalues (they are integers) and the eigenvectors of **(A9, D14, D17, D18, D19)**:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}.$$

10. Let A, B, and C be $n \times n$ matrices. Prove that $A(BC) = (AB)C$. **(D1)**