LINEAR ALGEBRA COMPREHENSIVE EXAM

Summer 2009, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Prove that a linear system of equations with two different solutions has an infinite number of solutions. **(A1, A6, C7)**

2. Express solutions of the system

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 3x_2 & + & 2x_3 & - & 3x_4 & = & 0 \\ 3x_1 & - & 5x_2 & + & 3x_3 & - & 4x_4 & = & 0 \\ -x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 0 \end{array}$$

as the translation of a vector space. **(A1, A2, A3, A7, A9, B4)**

3. Consider the inner product space $C_{-\pi,\pi}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

In the vector space $C_{-\pi,\pi}$ find the angle between $\cos x$ and $\sin x$. **(B8, B9, C15)**

4. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\|\vec{w}\|$ to prove the Triangle Inequality in an inner-product space. **(B8, B10, C15)**

5. Find the projection of $[1, 2, 1, 2]$ onto the plane $x + y + z + w = 0$. **(B3, B7, B8, C17, C19)**

6. Do each of the following:

(a) What is the vector space $\mathbb{R}^n$? **(B1)**

(b) What is the difference between a point in $\mathbb{R}^n$ and a vector in $\mathbb{R}^n$? **(B3)**

(c) What is a vector space isomorphism? **(C13)**

7. Consider the vector spaces $V = V' = \text{span}\{\cos x, \sin x\}$ with ordered bases $B = B' = \{\cos x, \sin x\}$. Let $T : V \to V'$ be defined as the differentiation operator. Find the matrix A that represents T relative to B, B'. **(C7, C8, C11, C15)**

8. Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Show that there is one and only one vector $\vec{p}$ in W such that $\vec{b} - \vec{p}$ is perpendicular to every vector in W. **(C4, C18, C19)**

9. Diagonalize $\begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$ and find A^{100}. **(D1, D17, D18, D20, D22)**

10. (a) What is an elementary matrix? **(D7)**

(b) Express A and A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$. **(D3, D7, D8, D9)**