LINEAR ALGEBRA COMPREHENSIVE EXAM  
Summer 2009, Prepared by Dr. Robert Gardner  
July 10, 2009

NAME ___________________________ STUDENT NUMBER _______________________

Be clear and give all details. Use symbols correctly (such as equal signs). The numbers in bold  
faced parentheses indicate the number of the topics covered in that problem from the Study Guide. 
No calculators! You may omit two problems. Indicate which two problems you are omitting: ____  
and ____. There is a three hour time limit.

1. Prove that a linear system of equations with two different solutions has an infinite number  
of solutions. (A1, A6, C7)

2. Express solutions of the system

\[
\begin{align*}
    x_1 - 2x_2 + x_3 - x_4 &= 0 \\
    2x_1 - 3x_2 + 2x_3 - 3x_4 &= 0 \\
    3x_1 - 5x_2 + 3x_3 - 4x_4 &= 0 \\
    -x_1 + x_2 - x_3 + 2x_4 &= 0
\end{align*}
\]

as the translation of a vector space. (A1, A2, A3, A7, A9, B4)

3. Consider the inner product space \( C_{-\pi, \pi} \) of continuous functions on \([-\pi, \pi]\) with the inner product  
of \( f \) and \( g \) defined as

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx.
\]

In the vector space \( C_{-\pi, \pi} \) find the angle between \( \cos x \) and \( \sin x \). (B8, B9, C15)

4. Use the Schwarz Inequality, which states that for vectors \( \vec{v} \) and \( \vec{w} \) in an inner-product space,  
we have \(|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\|\vec{w}\|\) to prove the Triangle Inequality in an inner-product space. (B8,  
B10, C15)

5. Find the projection of \([1, 2, 1, 2]\) onto the plane \( x + y + z + w = 0 \). (B3, B7, B8, C17, C19)

6. Do each of the following:

   (a) What is the vector space \( \mathbb{R}^n \)? (B1)

   (b) What is the difference between a point in \( \mathbb{R}^n \) and a vector in \( \mathbb{R}^n \)? (B3)

   (c) What is a vector space isomorphism? (C13)
7. Consider the vector spaces \( V = V' = \text{span}\{\cos x, \sin x\} \) with ordered bases \( B = B' = \{\cos x, \sin x\} \).
   Let \( T : V \to V' \) be defined as the differentiation operator. Find the matrix \( A \) that represents \( T \) relative to \( B, B' \). (C7, C8, C11, C15)

8. Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( \vec{b} \) be a vector in \( \mathbb{R}^n \). Show that there is one and only one vector \( \vec{p} \) in \( W \) such that \( \vec{b} - \vec{p} \) is perpendicular to every vector in \( W \). (C4, C18, C19)

9. Diagonalize \( \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix} \) and find \( A^{100} \). (D1, D17, D18, D20, D22)

10. (a) What is an elementary matrix? (D7)
    (b) Express \( A \) and \( A^{-1} \) as a product of elementary matrices where \( A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \). (D3, D7, D8, D9)