LINEAR ALGEBRA COMPREHENSIVE EXAM

Summer 2003, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit two numbered problems. Indicate which two problems you are omitting: _____ and _____. NO CALCULATORS!!! Time limit: 3 hours.

1. Consider the matrix

$$A = \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix}.$$

Put A in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. (**A3, A4, A5**)

2. State three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Does the system

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 8 & 3 \\ -1 & -3 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

have a unique solution (explain)? (**A5, 8, A9**)

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $[1,2,3]$ and $[4,5,6]$ and passes through the point $(7,8,9)$. Find the equation of the plane (in terms of x, y and z coordinates) and express the plane as a translation of a vector space. (**B4, B12**)

4. State the "Fundamental Theorem of Finite Dimensional Vector Spaces" (which deals with isomorphisms of vector spaces). Let V_1 and V_2 be vector spaces with real scalars. What conditions must be satisfied for $\pi : V_1 \to V_2$ to be an isomorphism? Are the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ isomorphic? Explain. (**B4, B12**)

5. Consider the vectors $\vec{v}_1 = x^2 - 2x + 1$, $\vec{v}_2 = 2x^2 + 5x + 11$, and $\vec{v}_3 = 3x^2 + 7x + 17$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain. (**C5, C11, C15**)

6. Find the projection of x onto $\sin x$ in the inner product space $C_{0,\pi}$ of continuous functions on $[0, \pi]$ with the inner product of f and g defined as

$$\langle f, g \rangle = \int_0^{\pi} f(x)g(x)\,dx.$$

(**C15, C17**)

7. Let A be an $n \times n$ matrix. Prove that the collection of all solutions to the equation $A\vec{x} = \vec{0}$ form a subspace of $\mathbb{R}^n$. (**A9, C2, C4**)

8. Express A and A^{-1} as products of elementary matrices where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(**D3, D7, D8, D9**)

9. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

10. Prove that for $A = [a_{ij}]$ and $B = [b_{ij}]$ $n \times n$ matrices, we have $(AB)^T = B^T A^T$. (**D1, D4**)

11. Consider

$$A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

Find the rank, a basis for the row space, and a basis for the column space. (**A4, A5, D6, D10**)

12. Diagonalize $A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$ and calculate A^{100}. (**D1, D17, D20**)