LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2015b, Prepared by Dr. Robert Gardner
May 1, 2015

NAME ____________________________ Start Time: _______ End Time: _______

Be clear and give all details. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators and turn off your cell phones! Use the paper provided and only write on one side. You may omit two problems. Indicate which two problems you are omitting: _____ and ____. There is a three hour time limit.

1. State three conditions on \( n \times n \) matrix \( A \) which would (each) imply that the system \( A \vec{x} = \vec{b} \) has a unique solution. Does the system
\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 8 & 3 \\
-1 & -3 & 0
\end{bmatrix}
\vec{x} = \vec{0}
\]
have a unique solution (explain)? (A5, A8, A9)

2. Consider the plane in \( \mathbb{R}^3 \) which contains the vectors \([1, 2, 3]\) and \([4, 5, 6]\) and passes through the point \((7, 8, 9)\). Find the equation of the plane (in terms of \( x, y \) and \( z \) coordinates) and express the plane as a translation of a vector space. (B4, B12)

3. Find the projection of \([1, 2, 1, 2]\) onto the plane \( x + y + z + w = 0 \). Explain your reasoning! (B3, B7, B8, C17, C19)

4. State the definition of vector space. Give an example of a vector space other than \( \mathbb{R}^n \) or \( \mathbb{C}^n \). (C1)

5. Transform the basis \([\{[1, 1, 1], [1, 0, 1], [0, 1, 1]\}]\) for \( \mathbb{R}^3 \) into an orthogonal basis using the Gram-Schmidt process. (C17, C19, C20, C21)

6. Consider the vectors \( \vec{v}_1 = x^2 + 2x + 3 \), \( \vec{v}_2 = 7x^2 - 5x + 2 \), and \( \vec{v}_3 = -4x^2 + 2x - 9 \) in \( \mathbb{P}_2 \), the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain. (C5, C11, C15)
7. Consider
\[
\begin{bmatrix}
0 & 6 & 6 & 3 \\
1 & 2 & 1 & 1 \\
4 & 1 & -3 & 4 \\
1 & 3 & 2 & 0
\end{bmatrix}.
\]
Find the rank, a basis for the row space, and a basis for the column space. (A4, A5, D6, D10)

8. (a) What is an elementary matrix? (D7)

(b) Express \( A \) and \( A^{-1} \) as products of elementary matrices where \( A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \). (D3, D7, D8, D9)

9. Prove that if \( \lambda \) is an eigenvalue of an \( n \times n \) matrix \( A \), then the set \( E_\lambda \) consisting of the zero vector together with all eigenvectors of \( A \) for this eigenvalue \( \lambda \) is a subspace of \( n \)-space. (C4, D17, D19)

10. Find the L/U decomposition of the matrix
\[
A = \begin{bmatrix}
1 & 3 & -1 \\
2 & 8 & 4 \\
-1 & 3 & 4
\end{bmatrix}.
\]
Explain your reasoning. (D23)