LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2016b, Prepared by Dr. Robert Gardner
April 22, 2016

NAME _______________________________ Start Time: _______ End Time: _______

Be clear and give all details. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide.

No calculators and turn off your cell phones! You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Express the solution of this system as a translation of a vector space:

\[ 3x_1 + 2x_2 + x_3 = 4 \]
\[ -5x_1 - 3x_2 - x_3 + x_4 = -7 \]
\[ 5x_1 + 4x_2 + 3x_3 + 2x_4 = 6 \]
\[ x_1 + x_2 + x_3 + x_4 = 1 \]

(A1, A2, A3, A7, B4)

2. Consider the matrix

\[ A = \begin{bmatrix} 1 & 14 & -4 & 7 \\ -3 & -6 & 0 & -9 \\ 5 & -8 & 6 & 9 \\ 2 & 13 & -3 & 9 \end{bmatrix} \]

Which columns contain pivots? What is a basis for the column space? (A3, A4, A5, D6)

3. Find the projection of \([1, 2, 1, 2]\) onto the plane \(x + y + z + w = 0\). Explain your reasoning! (B3, B7, B8, C17, C19)

4. State the definition of vector space. (C1)

5. Find the standard matrix representation (i.e. the representation with respect to the standard bases of \(\mathbb{R}^m\) and \(\mathbb{R}^n\)) of the linear transformation \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^4\) defined by \(T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3)\). (C7, C8)

6. Consider the vectors \(\vec{v}_1 = x^2 - 2x + 1\), \(\vec{v}_2 = 2x^2 + 5x + 11\), and \(\vec{v}_3 = 3x^2 + 7x + 17\) in \(\mathcal{P}_2\), the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. (C5, C11, C15)
7. Use the Schwarz Inequality, which states that for vectors \( \vec{v} \) and \( \vec{w} \) in an inner-product space, we have \( |\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\|\vec{w}\| \), to prove the Triangle Inequality in an inner-product space. (B8, B10, C15)

8. Find the eigenvalues (they are integers) and the eigenvectors of:

\[
A = \begin{bmatrix}
-2 & 0 & 0 \\
-5 & -2 & -5 \\
5 & 0 & 3
\end{bmatrix}.
\]

(A9, D14, D17, D18, D19)

9. Prove that if \( \lambda \) is an eigenvalue of an \( n \times n \) matrix \( A \), then the set \( E_\lambda \) consisting of the zero vector together with all eigenvectors of \( A \) for this eigenvalue \( \lambda \) is a subspace of \( n \)-space. (C4, D17, D19)

10. Find the L/U decomposition of the matrix

\[
A = \begin{bmatrix}
1 & 3 & -1 \\
2 & 8 & 4 \\
-1 & 3 & 4
\end{bmatrix}.
\]

Explain your reasoning. (D23)