LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2008, Prepared by Dr. Robert Gardner

April 25, 2008

NAME ____________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. (**A1, A7, B4**)

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that $A\vec{x} = \vec{b}$ has multiple solutions. (**A5, A8, A9**)

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$with the inner product of f and g defined as

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\, dx.$$

(**B8, B9, C15**)

4. Find the projection of vector $[1, 2, 3, 4]$ onto the line joining the points $(0, 4, -3, 2)$ and $(1, 4, 0, 2)$. (**B3, B7, B8, C17**)

5. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3)$. (**C7, C8**)

6. Do each of the following:

(a) What is the vector space $\mathbb{R}^n$? **(B1)**

(b) What is the difference between a point in $\mathbb{R}^n$ and a vector in $\mathbb{R}^n$? **(B3)**

(c) What is a vector space isomorphism? **(C13)**

7. Consider the vector spaces $V = V' = \text{span}\{\cos x, \sin x\}$ with ordered bases $B = B' = \{\cos x, \sin x\}$. Let $T : V \to V'$ be defined as the differentiation operator. Find the matrix A that represents T relative to B, B'. **(C7, C8, C11, C15)**

8. Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Show that there is one and only one vector $\vec{p}$ in W such that $\vec{b} - \vec{p}$ is perpendicular to every vector in W. **(C4, C18, C19)**

9. Diagonalize $\begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$ and find A^{100}. **(D1, D17, D18, D20, D22)**

10. (a) What is an elementary matrix? **(D7)**

(b) Express A and A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$. **(D3, D7, D8, D9)**