LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2008, Prepared by Dr. Robert Gardner
April 25, 2008

NAME ___________________________ STUDENT NUMBER ___________________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators!!! You may omit two numbered problems. Indicate which two problems you are omitting: _______ and _______.

1. Find the solution set of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. (A1, A7, B4)

2. Give three conditions on $n \times n$ matrix $A$ which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that $A\vec{x} = \vec{b}$ has multiple solutions. (A5, A8, A9)

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C_{0,2\pi}$ of continuous functions on $[0, 2\pi]$ with the inner product of $f$ and $g$ defined as

$$\langle f, g \rangle = \int_{0}^{2\pi} f(x)g(x) \, dx.$$  

(B8, B9, C15)

4. Find the projection of vector $[1, 2, 3, 4]$ onto the line joining the points $(0, 4, -3, 2)$ and $(1, 4, 0, 2)$. (B3, B7, B8, C17)

5. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3).$$  

(C7, C8)

6. Do each of the following:

(a) What is the vector space $\mathbb{R}^n$? (B1)

(b) What is the difference between a point in $\mathbb{R}^n$ and a vector in $\mathbb{R}^n$? (B3)

(c) What is a vector space isomorphism? (C13)
7. Consider the vector spaces \( V = V' = \text{span}\{\cos x, \sin x\} \) with ordered bases \( B = B' = \{\cos x, \sin x\} \). Let \( T : V \to V' \) be defined as the differentiation operator. Find the matrix \( A \) that represents \( T \) relative to \( B, B' \). (C7, C8, C11, C15)

8. Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( \vec{b} \) be a vector in \( \mathbb{R}^n \). Show that there is one and only one vector \( \vec{p} \) in \( W \) such that \( \vec{b} - \vec{p} \) is perpendicular to every vector in \( W \). (C4, C18, C19)

9. Diagonalize \[
\begin{bmatrix}
-3 & 5 \\
-2 & 4
\end{bmatrix}
\] and find \( A^{100} \). (D1, D17, D18, D20, D22)

10. (a) What is an elementary matrix? (D7)
    
    (b) Express \( A \) and \( A^{-1} \) as a product of elementary matrices where \( A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \). (D3, D7, D8, D9)