LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2007, Prepared by Dr. Robert Gardner April 27, 2007

NAME ______________________ STUDENT NUMBER ______________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators!!! You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of \( A \vec{x} = \vec{b} \) where
\[
A = \begin{bmatrix}
1 & -4 & 1 \\
3 & -13 & 0 \\
2 & -9 & -1
\end{bmatrix}
\quad \text{and} \quad
\vec{b} = \begin{bmatrix}
-2 \\
-10 \\
-8
\end{bmatrix}
\]
and express the solution as a translation of a vector space. (A1, A7, B4)

2. Give three conditions on \( n \times n \) matrix \( A \) which would (each) imply that the system \( A \vec{x} = \vec{b} \) has a unique solution. Give two conditions which would (each) imply that \( A \vec{x} = \vec{b} \) has multiple solutions. (A5, A8, A9)

3. Show that the vectors \( \sin x \) and \( \cos x \) are orthogonal in the inner product space \( C_{0,2\pi} \) of continuous functions on \([0,2\pi]\) with the inner product of \( f \) and \( g \) defined as
\[
\langle f, g \rangle = \int_{0}^{2\pi} f(x)g(x) \, dx.
\]
(B8, B9, C15)

4. Find the projection of vector \([1, 2, 3, 4]\) onto the line joining the points \((0, 4, -3, 2)\) and \((1, 4, 0, 2)\). (B3, B7, B8, C17)

5. Find the standard matrix representation (i.e. the representation with respect to the standard bases of \( \mathbb{R}^m \) and \( \mathbb{R}^n \)) of the linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) defined by
\[
T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3).
\]
(C7, C8)

6. Consider the space \( P_3 \) of all polynomials of degree 3 or less. Find the coordinate vector of \( x^3 + 3x^2 - 4x + 2 \) relative to the ordered basis \((x, x^2 - 1, x^3, 2x^2)\). (C11, C6, A1)

7. Transform the basis \{\((1, 0, 1), (0, 1, 2), (2, 1, 0)\)\} for \( \mathbb{R}^3 \) into an orthogonal basis using the Gram-Schmidt process. (C17, C19, C20, C21)

8. Express \( A \) and \( A^{-1} \) as products of elementary matrices where
\[
A = \begin{bmatrix}
2 & 9 \\
1 & 4
\end{bmatrix}.
\]
(D3, D7, D8, D9)
9. Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that the column space of $AC$ is contained in the column space of $A$. (D6, D10)

10. Find the eigenvalues (they are integers) and the eigenvectors of (A9, D14, D17, D18, D19):

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$