LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2004, Prepared by Dr. Robert Gardner

April 16, 2004

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit two numbered problems. Indicate which two problems you are omitting: _____ and _____. NO CALCULATORS!!! Time limit: 3 hours.

1. Consider the the matrix

$$A = \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix}.$$

Put A in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. **(A3, A4, A5)**

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 14 & -4 & 7 \\ -3 & -6 & 0 & -9 \\ 5 & -8 & 6 & 9 \\ 2 & 13 & -3 & 9 \end{bmatrix}.$$

Which columns contain pivots? What is a basis for the column space? **(A3, A4, A5, D6)**

3. Find the projection of $[1, 2, 3, 4]$ onto the plane $x + y + z + w = 0$. **(B3, B7, B8, C17, C19)**

4. State the definition of *vector space*. **(C1)**

5. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$, where $\mathcal{P}_3$ is the vector space of all polynomials of degree 3 or less, be defined by $T(p(x)) = D(p(x))$, the derivative of $p(x)$. Let the ordered basis for $\mathcal{P}_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix A which represents T relative to B, B'. **(C7, C8, C11, C15)**

6. Use the Gram-Schmidt process to find an orthonormal basis for the subspace $\text{span}\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\}$ of $\mathbb{R}^4$. **(C17, C19, C20, C21)**

7. Let A and C be matrices such that the product AC is defined. Prove that the column space of AC is contained in the column space of A. **(D6, D10)**

8. If A is an invertible $n \times n$ matrix, then $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$. Use this fact to find A^{-1} for

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

(D13, D15, D16)

9. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. **(C4, D17, D19)**

10. Consider

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}.$$

Put A in a row echelon form, keeping track of how each row operation affects the determinant, then calculate the determinant of A. **(A4, D15)**