LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2002, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Consider the matrix
$$A = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$
Put A in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. (**A3, A4, A5**)

2. State three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Does the system
$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 7 \\ 1 & 11 & 16 \end{bmatrix} \vec{x} = \vec{0}$$
have a unique solution (explain)? **(A5, A8, A9)**

3. Use Schwarz's Inequality, which states that for $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ we have $|\vec{v}_1 \cdot \vec{v}_2| \leq \|\vec{v}_1\|\|\vec{v}_2\|$ to prove the Triangle Inequality in $\mathbb{R}^n$. (**B8, B10**)

4. Consider the plane in $\mathbb{R}^3$ which contans the vectors $[1, 2, 3]$ and $[4, 5, 6]$ and passes through the point $(7, 8, 9)$. Find the equation of the plane (in terms of x, y and z coordinates) and express the plane as a translation of a vector space. **(B4, B12)**

5. State the "Fundamental Theorem of Finite Dimensional Vector Spaces" (which deals with isomorphisms of vector spaces). Let V_1 and V_2 be vector spaces with real scalars. What conditions must be satisfied for $\pi : V_1 \to V_2$ to be an isomorphism? Are the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ isomorphic? Explain. **(C12, C13, C14)**

6. State the definition of *vector space*. **(C1)**

7. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly dependent? Explain. **(C5, C11, C15)**

8. Find the projections of $\cos x$ onto $\sin x$ in the inner product space $C_{-\pi,\pi}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

HINT: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. **(C15, C17)**

9. Prove that for $A = [a_{ij}]$ and $B = [b_{ij}]$ $n \times n$ matrices, we have $(AB)^T = B^T A^T$. **(D1, D4)**

10. Consider

$$A = \begin{bmatrix} 2 & 3 & 4 & 6 \\ 2 & 0 & -9 & 6 \\ 4 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Put A in a row echelon form, keeping track of how each row operation affects the determinant. Then calculate the determinant of A. **(A4, D15)**

11. Find the eigenvalues (they are integers) and eigenvectors of

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}.$$

(A9, D14, D17, D18, D19)

12. Diagonalize $A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$ and calculate A^{100}. **(D1, D17, D20)**